

## Personal and contact information

<i>Name</i>	Armin Straub born December 15, 1983, in Offenbach a.M. (Germany)
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<i>Home address</i>	Ahornweg 52 63150 Heusenstamm (Germany)
<i>E-Mail</i>	<code>astraub@illinois.edu</code> <code>mail@arminstraub.com</code>
<i>Website</i>	<code>http://arminstraub.com</code> containing preprints, slides of talks, and a current CV
<i>Research Interests</i>	number theory, special functions, enumerative combinatorics symbolic and numerical computation

## Academic employment

<i>Jan 2014 – May 2016</i> <i>&amp; Aug 2012 – Dec 2012</i>	University of Illinois at Urbana-Champaign J. L. Doob Research Assistant Professor
<i>Jan 2013 – Dec 2013</i>	Max-Planck-Institut für Mathematik, Bonn (Germany) Postdoctoral fellow

## Academic education

<i>Aug 2008 – May 2012</i>	Ph.D. in Mathematics from Tulane University (USA) thesis: “Arithmetic aspects of random walks and methods in definite integration” advisor: Victor H. Moll ( <code>vhm@tulane.edu</code> ) co-advisor: Jonathan M. Borwein ( <code>jonathan.borwein@newcastle.edu.au</code> )
<i>Aug 2007 – Apr 2008</i>	Diplom in Mathematics from TU Darmstadt (Germany) thesis: “Local recognition of reflection graphs on Coxeter groups” grade: passed with distinction, supervisor: Ralf Köhl (né Gramlich)
<i>Aug 2006 – May 2007</i>	M.S. in Mathematics from Tulane University (USA)
<i>Apr 2003 – Aug 2006</i>	Student of Mathematics at TU Darmstadt (Germany) minor in Computer Science

## Preprints

- [5] *Multivariate Apéry numbers and supercongruences of rational functions*  
submitted, 2014
- [4] *Supercongruences for sporadic sequences*  
(with Robert Osburn, Brundaban Sahu)  
submitted, 2013
- [3] *Positivity of rational functions and their diagonals*  
(with Wadim Zudilin)  
submitted, 2013
- [2] *On a secant Dirichlet series and Eichler integrals of Eisenstein series*  
(with Bruce C. Berndt)  
submitted, 2013
- [1] *On multiple and infinite log-concavity*  
(with Luis A. Medina)  
submitted, 2013

## Refereed publications

- [24] *On lattice sums and Wigner limits*  
(with David Borwein, Jonathan M. Borwein)  
published in Journal of Mathematical Analysis and Applications, Vol. 414, Nr. 2, 2014, p. 489-513
- [23] *On gamma quotients and infinite products*  
(with Marc Chamberland)  
published in Advances in Applied Mathematics, Vol. 51, Nr. 5, 2013, p. 546-562
- [22] *Relations for Nielsen polylogarithms*  
(with Jonathan M. Borwein)  
to appear in Journal of Approximation Theory (special issue dedicated to Richard Askey), 2013
- [21] *The Zagier polynomials. Part II: Arithmetic properties of coefficients*  
(with Mark W. Coffey, Valerio De Angelis, Atul Dixit, Victor H. Moll, Christophe Vignat)  
to appear in The Ramanujan Journal, 2014
- [20] *A solution of Sun's \$520 challenge concerning  $\frac{520}{\pi}$*   
(with Mathew Rogers)  
published in International Journal of Number Theory, Vol. 9, Nr. 5, 2013, p. 1273-1288
- [19] *On formulas for  $\pi$  experimentally conjectured by Jauregui-Tsallis*  
(with Tewodros Amdeberhan, David Borwein, Jonathan M. Borwein)  
published in Journal of Mathematical Physics, Vol. 53, Nr. 7, 2012, p. 073708:1-15
- [18] *Mahler measures, short walks and log-sine integrals*  
(with Jonathan M. Borwein)  
published in Theoretical Computer Science (special issue on Symbolic and Numeric Computation), Vol. 479, Nr. 1, 2013, p. 4-21

- [17] *Log-sine evaluations of Mahler measures, part II*  
(with David Borwein, Jonathan M. Borwein, James Wan)  
published in *Integers* (Selfridge memorial volume), Vol. 12, Nr. 6, 2012, p. 1179-1212
- [16] *A sinc that sank*  
(with David Borwein, Jonathan M. Borwein)  
published in *American Mathematical Monthly*, Vol. 119, Nr. 7, Aug-Sep 2012, p. 535-549
- [15] *Special values of generalized log-sine integrals*  
(with Jonathan M. Borwein)  
published in *Proceedings of ISSAC 2011* (36th International Symposium on Symbolic and Algebraic Computation), ACM Press, Jun 2011, p. 43-50 — received *ISSAC 2011 Distinguished Student Author Award*
- [14] *Densities of short uniform random walks (with an appendix by Don Zagier)*  
(with Jonathan M. Borwein, James Wan, Wadim Zudilin)  
published in *Canadian Journal of Mathematics*, Vol. 64, Nr. 5, 2012, p. 961-990
- [13] *Ramanujan's Master Theorem*  
(with Tewodros Amdeberhan, Ivan Gonzalez, Marshall Harrison, Victor H. Moll)  
published in *The Ramanujan Journal*, Vol. 29, Nr. 1, 2012, p. 103-120
- [12] *Log-sine evaluations of Mahler measures*  
(with Jonathan M. Borwein)  
published in *Journal of the Australian Mathematical Society* (special issue dedicated to Alf van der Poorten), Vol. 92, Nr. 1, 2012, p. 15-36
- [11] *A q-analog of Ljunggren's binomial congruence*  
published in *DMTCS Proceedings: 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC)*, Jun 2011, p. 897-902
- [10] *The method of brackets. Part 2: Examples and applications*  
(with Ivan Gonzalez, Victor H. Moll)  
published in "Gems in Experimental Mathematics", *Contemporary Mathematics*, Vol. 517, 2010, p. 157-171
- [9] *Three-step and four-step random walk integrals*  
(with Jonathan M. Borwein, James Wan)  
published in *Experimental Mathematics*, Vol. 22, Nr. 1, 2013, p. 1-14
- [8] *Some arithmetic properties of short random walk integrals*  
(with Jonathan M. Borwein, Dirk Nuyens, James Wan)  
published in *The Ramanujan Journal*, Vol. 26, Nr. 1, 2011, p. 109-132
- [7] *Random walks in the plane*  
(with Jonathan M. Borwein, Dirk Nuyens, James Wan)  
published in *DMTCS Proceedings: 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC)*, Jul 2010, p. 155-166
- [6] *Wallis-Ramanujan-Schur-Feynman*  
(with Tewodros Amdeberhan, Olivier Espinosa, Victor H. Moll)  
published in *American Mathematical Monthly*, Vol. 117, Nr. 15, Aug 2010, p. 618-632

- [5] *Closed-form evaluation of integrals appearing in positronium decay*  
(with Tewodros Amdeberhan, Victor H. Moll)  
published in Journal of Mathematical Physics, Vol. 50, Nr. 10, Oct 2009, p. 103528:1-6
- [4] *A fast numerical algorithm for the integration of rational functions*  
(with Dante Manna, Luis Medina, Victor H. Moll)  
published in Numerische Mathematik, Vol. 115, Nr. 2, Apr 2010, p. 289-307
- [3] *The  $p$ -adic valuation of  $k$ -central binomial coefficients*  
(with Tewodros Amdeberhan, Victor H. Moll)  
published in Acta Arithmetica, Vol. 140, Nr. 1, 2009, p. 31-42
- [2] *The local recognition of reflection graphs of spherical Coxeter groups*  
(with Ralf Gramlich, Jonathan I. Hall)  
published in Journal of Algebraic Combinatorics, Vol. 32, Nr. 1, Aug 2010, p. 1-14
- [1] *Positivity of Szegő's rational function*  
published in Advances in Applied Mathematics, Vol. 41, Nr. 2, Aug 2008, p. 255-264

## Other (non-refereed) publications

- [3] *The integrals in Gradshteyn and Ryzhik. Part 22: Bessel-K functions*  
(with Larry Glasser, Karen T. Kohl, Christoph Koutschan, Victor H. Moll)  
published in Scientia, Series A: Math. Sciences 22, 2012, p. 129-151
- [2] *The integrals in Gradshteyn and Ryzhik. Part 9: Combinations of logarithms, rational and trigonometric functions*  
(with Tewodros Amdeberhan, Victor H. Moll, Jason Rosenberg, Pat Whitworth)  
published in Scientia, Series A: Math. Sciences 17, 2009, p. 27-44
- [1] *The integrals in Gradshteyn and Ryzhik. Part 8: Combinations of powers, exponentials and logarithms*  
(with Victor H. Moll, Jason Rosenberg, Pat Whitworth)  
published in Scientia, Series A: Math. Sciences 16, 2008, p. 41-50

## Research Talks

- |                     |  |
|---------------------|--|
| <i>Apr 13, 2014</i> | Multivariate Apéry numbers and supercongruences of rational functions<br>AMS Spring Central Sectional Meeting 2014, Special Session on Recent Developments in Number Theory, Texas Tech University |
| <i>Apr 3, 2014</i>  | Properties and applications of Apéry-like numbers<br>Number Theory Seminar, University of Illinois at Urbana-Champaign   |
| <i>Nov 18, 2013</i> | On the ubiquity of modular forms and Apéry-like numbers<br>Algebra and Number Theory Seminar, University College Dublin (IE)   |
| <i>Nov 12, 2013</i> | On a secant Dirichlet series and Eichler integrals of Eisenstein series<br>Number Theory Seminar, University of Cologne (DE)   |
| <i>Oct 17, 2013</i> | On the ubiquity of modular forms and Apéry-like numbers<br>Algebra and Combinatorics Seminar, Tulane University  |

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- Oct 12, 2013* On a secant Dirichlet series and Eichler integrals of Eisenstein series  
AMS Fall Eastern Sectional Meeting 2013, Special Session on Modular Forms and Modular Integrals in Memory of Marvin Knopp, Temple University
- Oct 9, 2013* On the ubiquity of modular forms and Apéry-like numbers  
Algorithmic Combinatorics Seminar, RISC, Johannes Kepler University (AT)
- Jul 10, 2013* A solution of Sun's \$520 challenge concerning  $520/\pi$   
SIAM Annual Meeting, Minisymposium on Symbolic Computation and Special Functions, San Diego
- Mar 14, 2013* A solution of Sun's \$520 challenge concerning  $520/\pi$   
27th Automorphic Forms Workshop, University College Dublin (IE)
- Feb 13, 2013* Arithmetic aspects of short random walks  
Number Theory Lunch Seminar, Max-Planck-Institut für Mathematik, Bonn (DE)
- Jan 29, 2013* Arithmetic aspects of short random walks  
Number Theory Seminar, University of Cologne (DE)
- Nov 15, 2012* On the  $q$ -binomial coefficients and binomial congruences  
 $q$ -Series Seminar, University of Illinois at Urbana-Champaign
- Oct 28, 2012* An application of modular forms to short random walks  
AMS Fall Western Sectional Meeting 2012, Special Session on Harmonic Maass Forms and  $q$ -Series, University of Arizona
- Oct 13, 2012* A  $q$ -analog of Ljunggren's binomial congruence  
Midwest Number Theory Conference for Graduate Students and Recent Ph.D IX, University of Illinois at Urbana-Champaign
- Sep 27, 2012* Arithmetic aspects of short random walks  
Number Theory Seminar, University of Illinois at Urbana-Champaign
- Aug 10, 2012* An application of modular forms to short random walks  
1st US-EU Conference on Automorphic Forms and Related Topics, RWTH Aachen (DE)
- Jan 7, 2012* Symbolic evaluation of log-sine integrals in polylogarithmic terms  
AMS Joint Mathematics Meetings 2012, Boston
- Oct 6, 2011* Hypergeometric evaluations of the densities of short random walks  
AG11—SIAM Conference on Applied Algebraic Geometry, Minisymposium on Symbolic Combinatorics, North Carolina State University
- Aug 24, 2011*  $q$ -binomial coefficient congruences  
CARMA Analysis and Number Theory Seminar, University of Newcastle (AU)
- Jun 9, 2011* Special values of generalized log-sine integrals  
ISSAC 2011—36th International Symposium on Symbolic and Algebraic Computation, San Jose — received *ISSAC 2011 Distinguished Student Author Award*
- May 19, 2011* Applications and evaluations of log-sine integrals  
JonFest 2011—Workshop on Computational and Analytical Mathematics in honour of Jonathan Borwein's 60th birthday, The IRMACS Centre, Simon Fraser University (CA)
- Jan 9, 2011* On the method of brackets  
AMS Joint Mathematics Meetings 2011, Special Session on Mathematics Related to Feynman Diagrams, New Orleans
- Oct 14, 2010* On infinite logconcavity  
Colloquium of the Mathematics Department, University of Newcastle (AU)

- Aug 2, 2010* Random walks in the plane  
FPSAC 2010 (Formal Power Series & Algebraic Combinatorics), San Francisco State University
- Aug 18, 2009* Random walk integrals  
CARMA Workshop on Multidimensional Numerical Integration and Special Function Evaluation, University of Newcastle (AU)

## Educational Talks

- Feb 20, 2014* An introduction to infinite log-concavity  
Graduate Student Number Theory Seminar, University of Illinois at Urbana-Champaign
- Oct 15, 2013* Tools for special functions and special numbers  
Graduate Student Colloquium of the Mathematics Department, Tulane University
- Apr 26, 2012* On the distance traveled in a few random steps  
GSSA Interdisciplinary Colloquium Series, Tulane University
- Mar 7, 2012* Pre  $\pi$  fest: A short portrayal of random facts  
Pi Day Pre-Game by Science and Engineering Honor Society (SEHS), Tulane University
- Oct 27, 2011* Random walks and where to find a drunkard  
Science and Engineering Honor Society (SEHS) Student Seminar, Tulane University
- Apr 12, 2011* How far does a drunkard get?  
Graduate Student Colloquium of the Mathematics Department, Tulane University
- Dec 10, 2007* Nonstandard analysis  
Student Colloquium (StuVo) of the Mathematics Department, TU Darmstadt (DE)

## Poster presentations

- Jun 13, 2011* A  $q$ -analog of Ljunggren's binomial congruence  
FPSAC 2011 (Formal Power Series & Algebraic Combinatorics), Reykjavik (IS)
- Aug 8, 2010* Random walk integrals  
School of Science and Engineering Research Day Poster Session, Tulane University — selected 1st place for best graduate student poster

## Academic visits

- Jul 30 – Aug 10, 2012* attended the summer school “Building Bridges: 1<sup>st</sup> US-EU Conference on Automorphic Forms and Related Topics”, RWTH Aachen (DE)
- Jul 28 – Aug 26, 2011* visited Jonathan M. Borwein at University of Newcastle (AU)
- Aug 9 – Nov 6, 2010* visited Jonathan M. Borwein at University of Newcastle (AU)
- Jun 21 – Jul 2, 2010* attended the summer graduate workshop “Sage Days 22: Computing with Elliptic Curves”, MSRI, Berkeley, CA
- Jul 30 – Aug 20, 2009* visited Jonathan M. Borwein at University of Newcastle (AU)
- Jun 1 – June 26, 2009* visited Marc Chamberland at Grinnell College
- Mar 2 – Mar 20, 2009* attended the workshop “Number Theory and Physics”, ESI (Erwin Schrödinger International Institute for Mathematical Physics), Vienna (AT)

## Academic services and memberships

<i>since 2012</i>	Reviewer for Mathematical Reviews (3 reviews)
<i>since 2007</i>	Referee for the following journals and proceedings: <ul style="list-style-type: none"> <li>• International Journal of Number Theory (4 times, 2011–2014)</li> <li>• Journal of Mathematical Analysis and Applications (4 times, 2012–2014)</li> <li>• Ars Combinatoria (2 times, 2013–2014)</li> <li>• Journal of Number Theory (2 times, 2013)</li> <li>• The Ramanujan Journal (8 times, 2010–2013)</li> <li>• The American Mathematical Monthly (2013)</li> <li>• Integers (2013)</li> <li>• Computer Physics Communications (2013)</li> <li>• Rocky Mountain Journal of Mathematics (2011)</li> <li>• Applied Mathematics Letters (2011)</li> <li>• AISC2010 (Artificial Intelligence and Symbolic Computation) (2010)</li> <li>• Contemporary Mathematics (2009)</li> <li>• Journal of Symbolic Computation (2007)</li> </ul>
<i>2009 – 2010</i>	Coorganizer of the Graduate Student Colloquium at Tulane University
<i>2008 – 2010</i>	GSSA (Graduate Studies Student Association) representative of the Mathematics Department, Tulane University
<i>since 2014</i>	Member of the American Mathematical Society (AMS)
<i>since 2013</i>	Member of the Society for Industrial and Applied Mathematics (SIAM)

## Academic awards

<i>May 2012</i>	“Excellent Graduate Student Teacher Award” Mathematics Department, Tulane University
<i>Jun 2011</i>	“ISSAC 2011 Distinguished Student Author Award” for the paper <i>Special values of generalized log-sine integrals</i> International Symposium on Symbolic and Algebraic Computation, San Jose
<i>May 2011</i>	“Excellence in Mathematics Graduate Student Award” Mathematics Department, Tulane University
<i>Apr 2010</i>	Poster “Random Walk Integrals” selected 1st place in Graduate Division School of Science and Engineering Research Day Poster Session, Tulane University
<i>May 2009</i>	2009-10 IBM Fellow in Computational Science Center for Computational Science, Tulane University
<i>Apr 2007</i>	“Outstanding First Year Graduate Student Award” Mathematics Department, Tulane University
<i>2006 – 2012</i>	Teaching assistantship, fellowship award, and tuition scholarship awarded by Tulane University for 2006/07, 2008/09, 2009/10, 2010/11, 2011/12

## Teaching experience

<i>Spring 2014</i>	Instructor for Introduction to Differential Equations Plus	two sections
<i>Fall 2012</i>	Instructor for Introduction to Differential Equations Plus	
	intro to differential equations, including systems of linear equations	
<i>Spring 2012</i>	Instructor for Real Analysis	300/600-level introduction to analysis
<i>Fall 2011</i>	Instructor for Calculus I	100-level calculus class
<i>Spring 2011</i>	Instructor for Statistics for Business	100-level statistics class
<i>Spring 2010</i>	Instructor for Statistics for Scientists	100-level statistics class
<i>Fall 2009</i>	Instructor for Calculus II	100-level calculus class
<i>Summer 2007</i>	Supervised undergraduate students at the research project “Experimental Mathematics” lead by Victor H. Moll, Tulane University	
<i>2006 – 2008</i>	Course Assistant for Calculus I, II, Linear Algebra, and Experimental Mathematics at Tulane University	
<i>2004 – 2008</i>	Course Assistant for Numerical Analysis, Linear Algebra, Statistics, Stochastic Analysis, and Algebra at TU Darmstadt	

## Other qualifications

<i>Languages</i>	German (native), English (fluent)
<i>Computer algebra</i>	Experience in several computer algebra systems including Mathematica, SAGE, GAP, Maple
<i>Programming skills</i>	Experience in various programming environments including Python, C++, PHP, SQL, HTML



# Math 286 – Differential Equations Plus

## Spring 2014

**Instructor.** Armin Straub

**Email.** [astraub@illinois.edu](mailto:astraub@illinois.edu)

**Course website.** <http://math.illinois.edu/~astraub/>

**Office.** 308 Altgeld

**Office phone.** (217) 300-0426 (please use e-mail whenever possible)

**Office hours.** Wednesdays, 10:30-11:30am, or by appointment

**Class schedule.**

Section E1 meets MTWR, 1:00-1:50pm, in 160 English Building.

Section G1 meets MTWR, 3:00-3:50pm, in 160 English Building.

**Grader Section E1.** Junxian Li ([jli135@illinois.edu](mailto:jli135@illinois.edu))

**Grader Section G1.** Liyuan Cao ([cao22@illinois.edu](mailto:cao22@illinois.edu))

**Tutor.** Nuoya Wang ([nwang12@illinois.edu](mailto:nwang12@illinois.edu))

**Tutoring hours.** There will be office hours for tutoring. Dates and time are posted on the course website.

## Course description

**Text.** *Differential Equations and Boundary Value Problems: Computing and Modelling*, 4th Edition, by C. H. Edwards and D. E. Penney (Prentice-Hall, 2008)

**Material covered.** roughly: 1.1–1.6, 2.1, 2.3, 3.1–3.8, 4.1, 5.1–5.6, 9.1–9.7, 10.1–10.3

## Grading

**Exams.** There will be three in-class midterm exams and a comprehensive final exam. Notes, books, calculators or computers are not allowed during any of the exams.

Our exam schedule is:

- Midterm Exam 1: Thursday, February 13 — 8:00-9:20pm, in 213 Greg Hall
- Midterm Exam 2: Thursday, March 13 — 8:00-9:20pm, in 23 Psych (Section E1) and 112 Chem Annex (Section G1)
- Midterm Exam 3: Thursday, April 17 — 8:00-9:20pm, in 150 ASL
- Final Exam: Friday, May 9 (tentative!)

There will be lectures held at the usual time on February 13, March 13 and April 17 to review exam material. Instead, there will be no class on April 14 as well as two other dates which will be announced later.

**Homework.** Homework will be assigned after most classes and is posted on our course website.

A designated part of the homework will be graded and is to be turned in on (or before) the Tuesday of the following week. To encourage you to do all homework, at least one problem will be chosen for each of the written exams (numbers used can be different). Homework needs to be turned in individually (though you are allowed to discuss it with classmates).

**Grades.** Your grade will be based on the total sum of your scores on three in-class midterm exams, homework, and the final exam.

- Midterm Exams: 55% in total (7% on weakest exam, 24% on each of the other two)
- Homework: 15% in total (lowest homework score dropped)
- Final Exam: 30%

## Further details

**Online grades.** Your grades will be posted to the Illinois Compass2g system which you can access at: <https://compass2g.illinois.edu>

Please report any discrepancies in recorded grades within two weeks.

**Online material.** This syllabus as well as relevant information and material for this course, including posted homework assignments, can be found at our course website.

**Grade scale.** For each exam and homework assignment you receive a numerical score rather than a letter grade. Your semester grade is guaranteed to be at least according to:

- 90-100: A-, A, A+
- 80-89: B-, B, B+
- 70-79: C-, C, C+
- 60-69: D-, D, D+
- 0-59: F

**Attendance.** Attendance of all lectures is expected and strongly recommended. Everything that is covered in class is relevant for the written exams.

**Make-up policy.** There will be no make-ups for missed homework assignments. Instead, the corresponding score will be dropped if appropriate documentation is presented. At the end of the semester, the lowest score on homework assignments will be dropped. If a midterm exam is missed, arrangements will be made on a case-by-case basis provided that appropriate documentation is presented.

**Dates of interest.**

[http://www.registrar.illinois.edu/registration/deadlines\\_SPRING2014.html](http://www.registrar.illinois.edu/registration/deadlines_SPRING2014.html)

- Monday, January 20: M. L. King Day
- Friday, March 14: Last day to drop without grade of W
- March 22–30: Spring Break
- Wednesday, May 7: Last day of instruction
- May 9–16: Final examination period

**Emergency information.** Please review carefully:

[http://www.math.illinois.edu/Bourbaki/emergency\\_syllabus\\_info.pdf](http://www.math.illinois.edu/Bourbaki/emergency_syllabus_info.pdf)

## Very basic examples of differential equations

**Example 1.** If  $y(x) = e^{x^2}$  then  $y'(x) = 2xe^{x^2} = 2xy(x)$ .

We say that  $y(x) = e^{x^2}$  is a **solution** to the **differential equation** (DE)  $y' = 2xy$ . ◇

**Example 2.** If  $y(x) = \sin(3x)$  then  $y'(x) = 3 \cos(3x) = 3\sqrt{1 - \sin^2(3x)}$ . Hence,  $y(x)$  solves the differential equation  $y'3\sqrt{1 - y(x)^2}$ .

On the other hand,  $y''(x) = -9\sin(3x) = -9y(x)$ . Thus,  $y(x)$  also solves the **second order** differential equation  $y'' = -9y$ . ◇

**Example 3.** Verify that  $e^y y' = 1$  is solved by  $y(x) = \ln(x + C)$ .

**Solution.**  $y'(x) = \frac{1}{x+C}$  and  $e^{y(x)} = x + C$ . Hence,  $e^y y' = 1$  indeed.

This means that  $y(x) = \ln(x + C)$  is a **one-parameter family** of solutions to the DE. ◇

**Example 4.** Consider the DE  $y'' = y' + 6y$ . For which  $r$  is  $e^{rx}$  a solution?

**Solution.** Plugging into the DE, we get  $r^2 e^{rx} = r e^{rx} + 6e^{rx}$  which we simplify to  $r^2 = r + 6$ . This has the two solutions  $r = -2$ ,  $r = 3$ . Hence  $e^{-2x}$  and  $e^{3x}$  are solutions of the DE.

In fact, we check that  $Ae^{-2x} + Be^{3x}$  is a two-parameter family of solutions to the DE. (It is no coincidence that the order of the DE is 2, whereas the previous example is order 1.) ◇

**Example 5.** Solve the DE  $y' = x^2 + x$ .

**Solution.**  $y(x) = \int (x^2 + x)dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$ . By the way, from Calculus, we know that there are no other solutions. In other words, we found the **general solution**.

To single out a particular solution, we can impose **initial values** such as  $y(0) = 1$ .

Solve  $y' = x^2 + x$  and  $y(0) = 1$ . This is called an **initial value problem** (IVP).

**Solution.**  $\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right]_{x=0} = C \stackrel{!}{=} 1$ . Hence,  $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$  is the (unique) solution of the IVP. ◇

**Review.** Verify that  $x(t) = \frac{1}{c-kt}$  is a one-parameter family of solutions to  $\frac{dx}{dt} = kx^2$ .

**Solution.**  $x'(t) = \frac{k}{(c-kt)^2} = kx(t)^2$

Solve the IVP:  $\frac{dx}{dt} = kx^2$ ,  $x(0) = 2$ . [What about  $x(0) = 0$ , instead?]

**Solution.**  $\left[\frac{1}{c-kt}\right]_{t=0} = \frac{1}{c} \stackrel{!}{=} 2$ . Hence  $c = \frac{1}{2}$ . Solution  $x(t) = \frac{1}{1/2-kt}$ . [Solution  $x(t) = 0$ . This may be seen as the case  $c \rightarrow \infty$ .]  $\diamond$

## Why care about DEs?

### A very simple model of population growth

If  $P(t)$  is the size of a population (eg. of bacteria) at time  $t$ , then the rate of change  $\frac{dP}{dt}$  might, from biological considerations, be (nearly) proportional to  $P(t)$ .

The corresponding **mathematical model** is described by the DE  $P' = kP$  where  $k$  is the constant of proportionality.

Mathematics (which we will soon have learned) tells us that the (only) solutions to this DE are given by  $P(t) = Ce^{kt}$  where  $C$  is some constant. (Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.)

**Example 6.** Suppose  $P(0) = 100$  and  $P(1) = 300$ . Find  $P(t)$ .

**Solution.**  $Ce^{k \cdot 0} = C = 100$  and  $Ce^k = 100e^k = 300$ . Hence  $k = \ln(3)$  and  $P(t) = 100e^{\ln(3)t} = 100 \cdot 3^t$ .  $\diamond$

Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

### The logistic model of population growth

If the population is constrained by resources, then  $P' = kP$  is not a good model. A model to take that into account is  $\frac{dP}{dt} = kP(1 - P/M)$ . This is the **logistic equation** ( $M$  is called the carrying capacity). Note that if  $P \ll M$  then  $1 - P/M \approx 1$  and we are back to the simpler model.

On the other hand, if  $P > M$  then  $1 - P/M < 0$  so that (assuming  $k > 0$ )  $P' < 0$  (i.e. population is shrinking).

**Example 7.** We will learn to solve such DEs. For now, we can still verify that

$$P(t) = \frac{CM e^{kt}}{M + C(e^{kt} - 1)}$$

solves the logistic DE. Note that  $P(0) = C$ . Also,  $\lim_{t \rightarrow \infty} P(t) = M$  (using, for instance, L'Hospital).  $\diamond$

### Movement of objects, velocity and acceleration

**Example 8.** A ball is dropped from a 100m tall building. How long until it reaches the ground? What's the speed when it hits the ground?

**Solution.** Let  $x(t)$  be the height at which the ball is at time  $t$ . Velocity is  $v(t) = x'(t)$  and acceleration  $a(t) = x''(t)$ . For a falling object,  $a(t) = -g$  (on earth, the gravitational acceleration is  $g \approx 9.8 \text{ m/s}^2$ ) and hence  $x(t)$  solves the DE  $x'' = -g$ . We also know the initial values  $x(0) = 100$ ,  $x'(0) = 0$ .

Hence,  $x'(t) = -gt$  and, therefore,  $x(t) = -\frac{1}{2}gt^2 + 100$ .

It reaches the ground when  $x(t) = -\frac{1}{2}gt^2 + 100 = 0$ , that is after  $t = \sqrt{200/g} \approx 4.5$  seconds.

The speed is  $|x'(4.5)| \approx 44.1 \text{ m/s}$ .  $\diamond$

## Understanding DEs without solving them

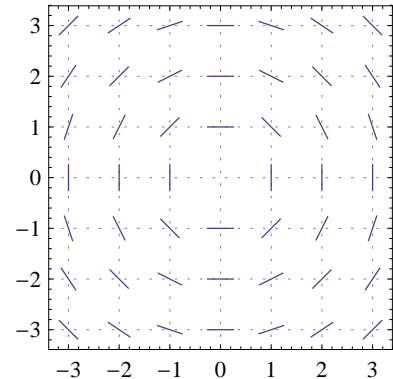
### Slope fields, or sketching solutions

**Example 9.** Consider the DE  $y' = -x/y$ .

Let's pick a point, say,  $(1, 2)$ . If a solution  $y(x)$  is passing through that point, then its slope has to be  $y' = -1/2$ . We therefore draw a small line through the point  $(1, 2)$  with slope  $-1/2$ . Continuing in this fashion for several other points, we obtain the **slope field** on the right.

With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether  $y(x) = \sqrt{r^2 - x^2}$  might indeed be a solution!

$$y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x). \text{ So, yes, we actually found solutions!} \quad \diamond$$



### Existence and uniqueness of solutions

**Definition 10.** A solution to the IVP  $y' = f(x, y)$ ,  $y(a) = b$  is a function  $y(x)$ , defined on an interval  $I$  containing  $a$ , such that  $y'(x) = f(x, y(x))$  for all  $x \in I$  and  $y(a) = b$ .

**Theorem 11.** <sup>1</sup>Consider the IVP  $y' = f(x, y)$ ,  $y(a) = b$ .

- (i) If  $f(x, y)$  is continuous [in a rectangle] around  $(a, b)$ , then there exists a (**local**<sup>2</sup>) solution.
- (ii) If both  $f(x, y)$  and  $\frac{\partial}{\partial y}f(x, y)$  are continuous [in a rectangle] around  $(a, b)$ , then there exists a (locally<sup>3</sup>) unique solution.

**Example 12.** Consider, again, the IVP  $y' = -x/y$ ,  $y(a) = b$ .

Here,  $f(x, y) = -x/y$  and  $\frac{\partial}{\partial y}f(x, y) = x/y^2$ . Both are continuous for all  $(x, y)$  with  $y \neq 0$ . Hence, if  $b \neq 0$  then the IVP has a unique solution.

Assume  $b > 0$  (things work similarly for  $b < 0$ ). Then  $y(x) = \sqrt{r^2 - x^2}$  solves the IVP if we choose  $r = \sqrt{a^2 + b^2}$ . Uniqueness means that there is no other solution to the IVP than this one (which we had guessed).  $\diamond$

**Example 13.** Discuss the IVP  $y' = y$ ,  $y(a) = b$ .

**Solution.** Here,  $f(x, y) = y$  and both  $f(x, y)$  and  $\frac{\partial}{\partial y}f(x, y)$  are continuous for all  $(x, y)$ . That means the IVP always has a unique solution (at least locally).

As a consequence, there can be no other solutions to the DE  $y' = y$  than the ones of the form  $y(x) = Ce^x$ . Why?! [Assume that  $y(x)$  satisfies  $y' = y$  and let  $(a, b)$  any value on the graph of  $y$ . Then  $y(x)$  solves the IVP  $y' = y$ ,  $y(a) = b$ ; but so does  $Ce^x$  with  $C = b/e^a$ . The uniqueness implies that  $y(x) = Ce^x$ .]  $\diamond$

**Example 14.** Last time, we verified that  $y' = y^2$ ,  $y(0) = 1$  is solved by  $y(x) = \frac{1}{1-x}$  on  $(-\infty, 1)$ .

Note that  $y(x)$  cannot be continuously extended past  $x = 1$ ; it is only a local solution! As in the previous example, we find that it is the unique solution to the IVP.  $\diamond$

1. The two parts of the theorem are famous results usually attributed to Peano and Picard–Lindelöf.
2. We call this a local solution at  $a$ , to emphasize that the interval  $I$  could be very small.
3. The interval in which the solution is unique could be smaller than the interval in which it exists.

**Review.** Existence and uniqueness of solutions ◇

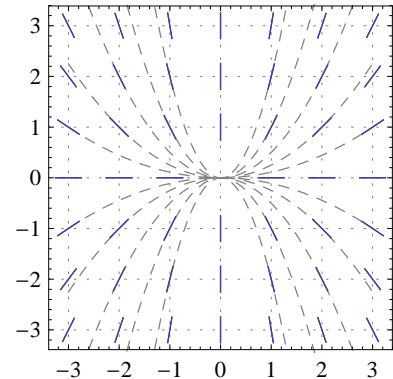
**Example 15.** Discuss the IVP  $xy' = 2y$ ,  $y(a) = b$ .

**Solution.** First, write as  $y' = f(x, y)$  with  $f(x, y) = 2y/x$ . We compute  $\frac{\partial}{\partial y} f(x, y) = 2/x$ .

Therefore, both  $f(x, y)$  and  $\frac{\partial}{\partial y} f(x, y)$  are continuous for all  $(x, y)$  with  $x \neq 0$ . Hence, if  $a \neq 0$  the IVP always has a (locally) unique solution.

We can verify that  $y(x) = Cx^2$  solves the DE. (compare slope field!)

This means that the IVP with  $y(0) = 0$  has infinitely many solutions. Since there are no other solutions (why?! look at slope field), the IVP with  $y(0) = b$  has no solution if  $b \neq 0$ . ◇



## Solving differential equations

### Separation of variables

**Example 16.** Solve the IVP  $xy' = 2y$ ,  $y(1) = -1$ .

**Solution.** Write as  $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}$ , multiply with  $dx$  and integrate both sides (this might be nonsense but remember that we can [and should] test whether we found a solution, so let's not worry) to get  $\int \frac{1}{y} dy = \int \frac{2}{x} dx$ . Hence,  $\ln |y| = 2 \ln |x| + C$ . Using  $y(1) = -1$ , we get  $C = 0$  and thus  $\ln(-y) = 2 \ln x$  (close to the initial value, we have  $|y| = -y$  and  $|x| = x$ ). Solving for  $y$ , we find  $y = -e^{2 \ln x} = -x^2$ . We easily verify that this is indeed a global solution (usually, it would only be a local solution and might have discontinuities such as poles). ◇

In general, [separation of variables](#) solves  $y' = g(x)h(y)$  by writing the DE as  $\frac{1}{h(y)} dy = g(x) dx$ .

Note that  $\frac{1}{h(y)} \frac{dy}{dx} = g(x)$  is indeed equivalent to  $\int \frac{1}{h(y)} dy = \int g(x) dx + C$ . Why?! (Apply  $\frac{d}{dx}$  to the integrals...)

**Example 17.**  $y' = x + y$  is a DE for which the variables can not be separated.

No worries, very soon we will have several tools to solve this DE as well. ◇

**Example 18.** Solve  $y' = ky^2$ .

**Solution.** Separate variables to get  $\frac{1}{y^2} \frac{dy}{dx} = k$ . Integrating  $\int \frac{1}{y^2} dy = \int k dx$ , we find  $-\frac{1}{y} = kx + C$  which we solve for  $y$  to get  $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$  (with  $D = -C$ ). That's the solution we verified in an earlier lecture!

Note that we did not find the solution  $y = 0$  (lost when dividing by  $y^2$ ). It is called a [singular solution](#) because it is not part of the [general solution](#) (the one-parameter family found above). ◇

**Remark 19.** We have to be careful about transforming our DE when using separation of variables: Just as the division by  $y^2$  made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE  $(y-1)y' = (y-1)ky^2$  has the same solutions as  $y' = ky^2$  plus the additional solution  $y = 1$  (which does not solve  $y' = ky^2$ ). ◇

**Example 20.** Solve  $y' = ky$ .

**Solution.** Write as  $\frac{dy}{dx} = ky$ , then  $\frac{1}{y}dy = kdx$  (note that we just lost the solution  $y = 0$ ).

Integrating gives  $\ln|y| = kx + C$ , hence  $|y| = e^{kx+C}$ . Since the RHS is never zero,  $y = \pm e^{kx+C} = De^{kx}$  (with  $D = \pm e^C$ ). Finally, note that  $D = 0$  corresponds to the singular solution  $y = 0$ . In summary, the general solution is  $De^{kx}$  with  $D \in \mathbb{R}$ .  $\diamond$

**Example 21.** Solve the IVP  $y' = -\frac{x}{y}$ ,  $y(0) = -3$ .

Last time: unique solution guaranteed *a priori*.

**Solution.** Separate variables to get  $ydy = -xdx$ . Integrating gives  $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ , hence  $x^2 + y^2 = D$  (with  $D = 2C$ ). Using  $y(0) = -3$ , we find  $0^2 + (-3)^2 = D$ . Thus,  $x^2 + y^2 = 9$  is an **implicit** form of the solution. In this case, we can solve for  $y$  to get  $y = -\sqrt{9 - x^2}$ .  $\diamond$

## Linear first-order equations

**Example 22.** Solve  $\frac{dy}{dx} = 2xy^2$ .

**Solution.**  $\frac{1}{y^2} \frac{dy}{dx} = 2x$ ,  $-\frac{1}{y} = x^2 + C$ . Hence the general solution is  $y = \frac{1}{D - x^2}$ . There is also the singular solution  $y = 0$ .

**Solution.** Note that  $\frac{1}{y^2} \frac{dy}{dx} = 2x$  can be written as  $\frac{d}{dx} \left[ -\frac{1}{y} \right] = \frac{d}{dx} [x^2]$ . Hence,  $-\frac{1}{y} = x^2 + C$ .

We now use the idea behind the second solution to solve other DEs.

The multiplication by  $\frac{1}{y^2}$  will be replaced by multiplication with the so-called “integrating factor”.  $\diamond$

**Example 23.** Solve  $y' = x - y$ .

(Note that we cannot use separation of variables.)

**Solution.**  $y' + y = x$ , now multiply with  $e^x$  (we will see in a moment, how to find this “integrating factor”).

Then  $e^x y' + e^x y = \frac{d}{dx} [e^x y]$ . On the other hand,  $xe^x = \frac{d}{dx} [xe^x - e^x]$ .

$\frac{d}{dx} [e^x y] = \frac{d}{dx} [xe^x - e^x]$  is equivalent to  $e^x y = xe^x - e^x + C$ . Hence,  $y = x - 1 + Ce^{-x}$ .  $\diamond$

In general, we can solve any **linear first-order equation**  $y' + P(x)y = Q(x)$  in this way.

- We want to multiply with an **integrating factor**  $f(x)$  such that the LHS of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{d}{dx} [f(x)y].$$

Since  $\frac{d}{dx} [f(x)y] = f(x)y' + f'(x)y$ , we need  $f'(x) = f(x)P(x)$  for that.

- An  $f(x)$  with that property is  $f(x) = e^{\int P(x)dx}$ . (Check!)
- The RHS of the DE only depends on  $x$ . It can be written as  $f(x)Q(x) = \frac{d}{dx} [\int f(x)Q(x)dx]$ .
- Hence, another way to write the DE is  $\frac{d}{dx} [f(x)y] = \frac{d}{dx} [\int f(x)Q(x)dx]$ .
- This shows that  $f(x)y = \int f(x)Q(x)dx + C$ , which means we have found the general solution (only need to divide by  $f(x)$ ).
- Note that this solution exists on any interval on which  $P$  and  $Q$  are continuous. (This is better than what Theorem 11 can predict.)

**Example 24.** Solve  $x^2 y' = 1 - xy + 2x$ ,  $y(1) = 3$ .

**Solution.** Write as  $\frac{dy}{dx} + P(x)y = Q(x)$  with  $P(x) = \frac{1}{x}$  and  $Q(x) = \frac{1}{x^2} + \frac{2}{x}$ . Integrating factor  $f(x) = e^{\ln x} = x$  (why do we write  $\ln x$  instead of  $\ln|x|$ ?). Hence,  $xy = \int (\frac{1}{x} + 2)dx = \ln x + 2x + C$ . Using  $y(1) = 3$ , we find  $C = 1$ . Solution  $y = \frac{\ln(x) + 2x + 1}{x}$ .  $\diamond$

**Review.** linear first-order equations

◇

**Example 25.** A tank contains 20gal of pure water. It is filled with brine (containing 2lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after  $t$  minutes?

**Solution.** volume (in gal) in tank after time  $t$  (in min):  $V(t) = 20 + (3 - 2)t = 20 + t$

amount of salt (in lb) in tank:  $x(t)$

concentration of salt (in lb/gal) in tank:  $\frac{x(t)}{V(t)}$

In the time interval  $[t, t + \Delta t]$ ,  $\Delta x \approx 3 \cdot 2 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$ .

Hence,  $x$  solves the IVP  $\frac{dx}{dt} = 6 - 2 \cdot \frac{x}{20+t}$ ,  $x(0) = 0$ .

This is a linear DE!

$\frac{dx}{dt} + \frac{2}{20+t}x = 6$ . The integrating factor is  $f(t) = e^{\int \frac{2}{20+t} dt} = (20+t)^2$ .

$(20+t)^2 \frac{dx}{dt} + 2(20+t)x = \frac{d}{dt}[(20+t)^2 x] = \frac{d}{dt}[\int 6(20+t)^2 dt] = \frac{d}{dt}[2(20+t)^3]$

Hence,  $(20+t)^2 x = 2(20+t)^3 + C$ . Using  $x(0) = 0$ , we find  $C = -2 \cdot 20^3$ .

This means that, after  $t$  minutes, the tank contains  $x(t) = 2(20+t) - \frac{2 \cdot 20^3}{(20+t)^2}$  pounds of salt.

As a consequence, we get that  $x(t) \approx 2(20+t) = 2V(t)$  for large  $t$ . Why does that make perfect sense?!

◇

## Substitutions

**Example 26.** Solve  $\frac{dy}{dx} = (x+y)^2$ .

**Solution.** Set  $u = x + y$ . Then  $\frac{du}{dx} = 1 + \frac{dy}{dx}$  and, hence,  $\frac{du}{dx} - 1 = u^2$ .

This DE for  $u$  can be solved by separation of variables:  $\frac{1}{1+u^2} du = dx$ ,  $\arctan(u) = x + C$ ,  $u = \tan(x + C)$ .

The solution of the original DE is  $y = u - x = \tan(x + C) - x$ .

◇

## Useful substitutions

Some important, easy-to-spot, cases:

- $y' = F\left(\frac{y}{x}\right)$ . This is called a **homogeneous equation**.  
Set  $u = \frac{y}{x}$ . Then  $y = ux$  and  $\frac{dy}{dx} = x \frac{du}{dx} + u$ . We get  $x \frac{du}{dx} + u = F(u)$ . This is always separable.
- $F(y'', y', x) = 0$ . 2nd order with “ $y$  missing”.  
Set  $u = y' = \frac{dy}{dx}$ . Then  $y'' = \frac{du}{dx}$ . We get the first-order DE  $F\left(\frac{du}{dx}, u, x\right) = 0$ .
- $F(y'', y', y) = 0$ . 2nd order with “ $x$  missing”.  
Set  $u = y' = \frac{dy}{dx}$ . Then  $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$ . We get the first-order DE  $F\left(u \frac{du}{dy}, u, y\right) = 0$ .

Examples follow next time...



**Review.** Useful substitutions ◇

**Example 27.** Solve  $(x - y)y' = x + y$ .

**Solution.** Divide the DE by  $x$  to get  $(1 - \frac{y}{x})y' = 1 + \frac{y}{x}$ . This is a homogeneous equation!

We therefore substitute  $u = \frac{y}{x}$ , to find the new DE  $xu' + u = \frac{1+u}{1-u}$ ,  $xu' = \frac{1+u^2}{1-u}$ .

Separation of variable:  $\frac{1-u}{1+u^2} du = \frac{1}{x} dx$ ,  $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$ .

Setting  $u = y/x$ , we get the (general) implicit solution  $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$ . ◇

**Example 28.** Solve  $y'' = x - y'$ .

**Solution.** Substitute  $u = y'$ . Then  $u' = x - u$ , which is linear with  $u = x - 1 + Ce^{-x}$ .

Hence  $y = \frac{1}{2}x^2 - x - Ce^{-x} + D$ . ◇

**Example 29.** Find a general solution to  $y'' = 2yy'$ .

**Solution.** We substitute  $u = y' = \frac{dy}{dx}$ . Then  $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$ .

Therefore, our DE turns into  $u \frac{du}{dy} = 2yu$ .

Dividing by  $u$ , we get  $\frac{du}{dy} = 2y$ . [Note that we loose the solution  $u = 0$ , which gives the singular solution  $y = C$ .]

Hence,  $u = y^2 + C$ . It remains to solve  $y' = y^2 + C$ . This is a separable DE.

$\frac{1}{C + y^2} dy = dx$ . Let us restrict to  $C = D^2 \geq 0$  here. (This means we will only find “half” of the solutions.)

$\int \frac{1}{D^2 + y^2} dy = \frac{1}{D^2} \int \frac{1}{1 + (y/D)^2} dy = \frac{1}{D} \arctan(y/D) = x + A$ .

Solving for  $y$ , we find  $y = D \tan(Dx + AD) = D \tan(Dx + B)$ . [B = AD] ◇

## Some more applications

### Acceleration–velocity models

We consider a falling object, and let  $y(t)$  be its height at time  $t$ .

If we only take earth’s gravitation into account, then the fall is modelled by  $y''(t) = -g$ .

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think. Reasonable physical assumptions imply that the resistance is proportional to the square of the velocity<sup>4</sup>. However, for “relatively slowly” falling objects one might empirically observe that the resistance is proportional to the velocity itself<sup>5</sup>. Or anything in between...

**Example 30.** If air resistance is proportional to velocity, then  $y''(t) = -\rho y'(t) - g$ . Solve this equation with initial conditions  $y(0) = y_0$ ,  $y'(0) = v_0$ . [Note that  $-\rho y' > 0$  because  $y' < 0$ .]

**Solution.** Set  $v = y'$ , get  $v' + \rho v = -g$  (linear!). Integrating factor  $e^{\rho t}$ .  $e^{\rho t}v = \int -ge^{\rho t} dt = -g/\rho e^{\rho t} + C$ .  
 $v(0) = y'(0) = v_0$  implies  $C = v_0 + g/\rho$ . Hence,  $v(t) = (v_0 + g/\rho)e^{-\rho t} - g/\rho$ .

$y(t) = \int v(t) dt = \dots = -(v_0/\rho + g/\rho^2)(e^{-\rho t} - 1) - gt/\rho + y_0$ , where we used that  $y(0) = y_0$ .

Note that  $\lim_{t \rightarrow \infty} v(t) = -g/\rho$ . **Terminal speed**  $g/\rho$ . (Important for idea behind a parachute!) ◇

4. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

5. Maybe it helps to imagine that, at slow speed, the falling object doesn’t exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

## Population models

To model a population, let  $P(t)$  be its size at time  $t$ .

$\beta(t)$ ,  $\delta(t)$ : birth and death rate [# of births/deaths (per unit of population per unit of time) at time  $t$ ]

$$\Delta P = \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t$$

$$\frac{dP}{dt} = (\beta(t) - \delta(t))P$$

**Example 31.** Some assumptions and corresponding models. [We'll come back here next class!]

- **(basic)** If  $\beta(t)$  and  $\delta(t)$  are constant, we get the exponential model  $\frac{dP}{dt} = kP$ .  $P(t) = Ce^{kt}$ .
- **(limited supply)**  $\delta(t)$  constant,  $\beta(t) = \beta_0 - \beta_1 P$   
 $\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta)P = aP - bP^2 = kP(1 - P/M)$ . This is the **logistic equation** from Lecture 2.
- **(rare species)**  $\delta(t)$  constant,  $\beta(t)$  proportional to  $P(t)$   
 $\frac{dP}{dt} = (\gamma P - \delta)P$ . The logistic equation, again.
- **(rare species with very long life)**  $\delta(t) = 0$ ,  $\beta(t)$  proportional to  $P(t)$   
 $\frac{dP}{dt} = kP^2$ . Solutions are  $P(t) = \frac{1}{C - kt}$  where  $P(0) = 1/C$ .  
 This explodes when  $t \rightarrow C/k$ . (But by then the species is not exactly rare anymore...)
- **(harvesting)** Each unit of time,  $h$  population units are harvested.  
 $\frac{dP}{dt} = (\beta(t) - \delta(t))P - h$   
 For instance,  $\frac{dP}{dt} = kP - h$  has  $P(t) = Ce^{kt} + h/k$ .
- **(spread of incurable virus)** Let  $P(t)$  count the number of infected population units among total of  $M$ .  
 $\delta(t) = 0$ ,  $\beta(t)$  proportional to  $M - P$   
 $\frac{dP}{dt} = kP(M - P)$ . Once again, the logistic equation. ◇

**Example 32.** Solve the **logistic equation**  $P' = kP(1 - P/M)$ . Separable!

**Solution.**  $\frac{-M}{P(P-M)} dP = \left(\frac{1}{P} - \frac{1}{P-M}\right) dP = k dt$ . We get  $\ln|P| - \ln|P-M| = \ln\left|\frac{P}{P-M}\right| = kt + C$ .

Hence,  $\frac{P}{P-M} = De^{kt}$  with  $D = \pm e^C$ . Thus  $P(t) = \frac{MDe^{kt}}{De^{kt} - 1}$ . [cf. Example 7.] ◇

## Linear higher-order differential equations

A **linear DE** is of the form  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$ .

- Let  $I$  be an interval on which  $p_j(x)$  and  $f(x)$  are continuous. If  $a \in I$  then a solution to the IVP with  $y(a) = b_0$ ,  $y'(a) = b_1$ , ...,  $y^{(n-1)}(a) = b_{n-1}$  always **exists** (actually, on all of  $I$ !) and is **unique**.

If  $f(x) = 0$ , then this is called a **homogeneous linear DE**. In that case:

- If  $y_1$  and  $y_2$  are solutions, then the **superposition**  $Ay_1 + By_2$  is a solution.
- **(general solution)** There are  $n$  solutions  $y_1, y_2, \dots, y_n$ , such that every solution is of the form  $C_1y_1 + \dots + C_ny_n$ . [These  $n$  solutions necessarily are, what we will call, **independent**.]

**Example 33.** Suppose that  $y_1$  and  $y_2$  solve  $y'' + p_1(x)y' + p_0(x)y = 0$ .

$(y_1 + y_2)'' + p_1(x)(y_1 + y_2)' + p_0(x)(y_1 + y_2) = \{y_1'' + p_1(x)y_1' + p_0(x)y_1\} + \{y_2'' + p_1(x)y_2' + p_0(x)y_2\} = 0 + 0$   
 In other words,  $y_1 + y_2$  is another solution of the DE. ◇

**Example 34.**  $x^2y'' + 2xy' - 6y = 0$  has solutions  $y_1 = x^2$ ,  $y_2 = x^{-3}$ .

Solve the IVP with  $y(2) = 10$ ,  $y'(2) = 15$ .

**Solution.** The general solution is  $y(x) = Ax^2 + Bx^{-3}$ .  $y'(x) = 2Ax - 3Bx^{-4}$ .

$y(2) = 4A + B/8 = 10$ ,  $y'(2) = 4A - 3/16B = 15$  has solutions  $A = 3$ ,  $B = -16$ . So  $y(x) = 3x^2 - 16/x^3$ . ◇

**Review.** population models

◇

**Example 35.** Short outbreaks of diseases among a population of constant size  $N$ .

Model the population as consisting of  $S(t)$  susceptible,  $I(t)$  infected and  $R(t)$  recovered individuals ( $N = S(t) + I(t) + R(t)$ ). In the SIR model,

$$\frac{dR}{dt} = \gamma I, \quad \frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I,$$

with  $\gamma$  modeling the recovery rate and  $\beta$  the infection rate. This is a [system of differential equations](#), something we will study in due time.

By the way, the following variation

$$\frac{dR}{dt} = \gamma I R, \quad \frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \gamma I R,$$

which assumes “[infectious recovery](#)”, was recently used to predict that facebook might loose 80% of its users by 2017. It’s that claim, not mathematics (or even the modeling), which attracted a lot of media attention<sup>6</sup>. ◇

## Homogeneous linear DEs with constant coefficients

**Review.** linear higher-order differential equations

◇

**Example 36.** Find the general solution of  $y'' - y' - 2y = 0$ .

*Hint:* Try solutions  $e^{rx}$ .

**Solution.** Plugging  $e^{rx}$  into the DE, we get  $r^2 e^{rx} - r e^{rx} - 2 e^{rx} = 0$ .

Equivalently,  $r^2 - r - 2 = 0$ . This is called the [characteristic equation](#). Its solutions are  $r = 2, -1$ .

This means we found the two solutions  $y_1 = e^{2x}$ ,  $y_2 = e^{-x}$ .

By superposition, the general solution<sup>7</sup> is  $y = A e^{2x} + B e^{-x}$ . ◇

This approach applies to any [homogeneous linear DE with constant coefficients](#)! If the characteristic equation has enough different roots, then we find the general solution. [by superposition!]

**Example 37.** Find the general solution of  $y''' + 7y'' + 14y' + 8y = 0$ .

**Solution.** The characteristic equation is  $r^3 + 7r^2 + 14r + 8 = (r + 1)(r + 2)(r + 4)$ .

Hence, we found the solutions  $y_1 = e^{-x}$ ,  $y_2 = e^{-2x}$ ,  $y_3 = e^{-4x}$ . That’s enough (independent!) solutions for a third-order DE. By superposition, the general solution is  $y(x) = A e^{-x} + B e^{-2x} + C e^{-4x}$ . ◇

6. <http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/>

7. Well, this is certainly a two-parameter family of solutions. To see that there can be no other solutions, you can convince yourself that for any choice of initial values  $y(a) = b_0$ ,  $y'(a) = b_1$  there exist values of  $A$  and  $B$  such that  $y = A e^{2x} + B e^{-x}$  takes these initial values. By uniqueness, this means there can be no other solutions. We will soon discuss this issue more closely.

**Example 38.** Solve the IVP  $y''' + 7y'' + 14y' + 8y = 0$  with  $y(0) = 1$ ,  $y'(0) = 0$ ,  $y''(0) = 1$ .

**Solution.** Last time, we found that the DE has the general solution  $y(x) = Ae^{-x} + Be^{-2x} + Ce^{-4x}$ .

$$y(x) = Ae^{-x} + Be^{-2x} + Ce^{-4x}, \quad y(0) = A + B + C = 1$$

$$y'(x) = -Ae^{-x} - 2Be^{-2x} - 4Ce^{-4x}, \quad y'(0) = -A - 2B - 4C = 0$$

$$y''(x) = Ae^{-x} + 4Be^{-2x} + 16Ce^{-4x}, \quad y''(0) = A + 4B + 16C = 1$$

Solving the system of linear equations, we find  $A = 3$ ,  $B = -5/2$ ,  $C = 1/2$ . Hence, the solution to the IVP is  $y(x) = 3e^{-x} - 5/2e^{-2x} + 1/2e^{-4x}$ .  $\diamond$

**Example 39.** Consider the IVP from the previous example.

Note that the DE let's us determine  $y'''(0) = -7y''(0) - 14y'(0) - 8y(0) = -15$  (without solving it!). By applying  $\frac{d}{dx}$  to the DE, we can likewise find  $y^{(4)}(0)$ ,  $y^{(5)}(0)$ , ...

This can be done with any DE and gives another indication why an IVP "usually" has a unique solution, and why initial conditions of this form are very natural to consider.  $\diamond$

**Example 40.** Find the general solution of  $y'' = 0$ . [Then,  $y^{(n)} = 0$ .]

**Solution.** We know from Calculus that the general solution is  $y(x) = A + Bx$ .

**Solution.** The characteristic equation is  $r^2 = 0$ . So one solution is  $y_1 = e^{0x} = 1$ . But what is a second solution? As Calculus showed, a second solution is  $y_2 = xe^{0x} = x$ . It turns out that this always works!  $\diamond$

**Example 41.** Find the general solution of  $y'' - 2y' + y = 0$ .

**Solution.** The characteristic equation is  $r^2 - 2r + 1 = (r - 1)^2$ . Hence,  $y_1 = e^x$ .

But what is the second solution? Inspired by the previous example, we can check that  $y_2 = xe^x$  is a solution.

Hence, the general solution is  $y(x) = Ae^x + Bxe^x$ .  $\diamond$

**Theorem 42.** Consider a **homogeneous linear DE with constant coefficients**  $y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0$ . (Its characteristic polynomial is  $p(r) = r^n + a_{n-1}r^{n-1} + \dots + a_1r + a_0$ .)

- If  $r_0$  is a root of the characteristic polynomial and if  $k$  is its multiplicity (this means that  $(r - r_0)^k$  is a factor of  $p(r)$ ), then  $e^{r_0x}$ ,  $xe^{r_0x}$ , ...,  $x^{k-1}e^{r_0x}$  are solutions of the DE.
- Combining these solutions for all roots  $r_0$ , actually gives the general solution.

This is because a polynomial of degree  $n$  has (counting with multiplicity) exactly  $n$  (possibly **complex**) roots. More on complex number in due time.

**Proof.** Set  $D = \frac{d}{dx}$ . A homogeneous linear DE with constant coefficients can be written as  $p(D)y = 0$ , where  $p(D)$  is a polynomial in  $D$ . [For instance,  $y'' - 2y' + y = 0$  is  $D^2y - 2Dy + y = (D^2 - 2D + 1)y = (D - 1)^2y = 0$ .]

In fact, we see that  $p(r)$  is just the characteristic polynomial!

If  $r_0$  is a root of the characteristic polynomial, then  $p(r) = q(r)(r - r_0)^k$ , where  $k \geq 1$  is its multiplicity.

The DE factors likewise and can be written as  $q(D)(D - r_0)^ky = 0$ .

From here we see that solutions to  $(D - r_0)^ky = 0$  will solve our original DE.

Let  $y(x)$  be a solution of  $(D - r_0)^ky = 0$ . Write it as  $y(x) = u(x)e^{r_0x}$  (we can always do that for some  $u(x)$ ).

Let  $u(x)$  be some function. Note that  $(D - r_0)[ue^{r_0x}] = u'e^{r_0x} + ur_0e^{r_0x} - r_0[ue^{r_0x}] = u'e^{r_0x}$ .

Repeating, we get  $(D - r_0)^2[ue^{r_0x}] = (D - r_0)[u'e^{r_0x}] = u''e^{r_0x}$  and, eventually,  $(D - r_0)^k[ue^{r_0x}] = u^{(k)}e^{r_0x}$ . In particular,  $(D - r_0)^ky = 0$  is solved by  $y = ue^{r_0x}$  if  $u^{(k)} = 0$ .

This latter condition gives  $u(x) = C_0 + C_1x + \dots + C_{k-1}x^{k-1}$  and it follows that  $y(x) = (C_0 + C_1x + \dots + C_{k-1}x^{k-1})e^{r_0x}$  solves our original DE, as claimed.  $\square$

**Review.** homogeneous linear DEs with constant coefficients ◇

**Example 43.** Find the general solution of  $y''' - y'' - 5y' - 3y = 0$ .

**Solution.** The characteristic equation is  $r^3 - r^2 - 5r - 3 = (r - 3)(r + 1)^2$ .

This corresponds to the solutions  $y_1 = e^{3x}$ ,  $y_2 = e^{-x}$ ,  $y_3 = x e^{-x}$ .

Hence, the general solution is  $y(x) = A e^{3x} + (B + Cx) e^{-x}$ . ◇

## Complex roots and complex exponentials

**Example 44.** Find the general solution of  $y'' + y = 0$ .

**Solution.** The characteristic equation is  $r^2 + 1 = 0$  which has no solutions over the reals.

Over the **complex numbers**, by definition, the roots are  $i$  and  $-i$ .

So the general solutions is  $y(x) = A e^{ix} + B e^{-ix}$ .

**Solution.** On the other hand, we easily check that  $y_1 = \cos(x)$  and  $y_2 = \sin(x)$  are two solutions.

Hence, the general solution can also be written as  $y(x) = C \cos(x) + D \sin(x)$ . ◇

**Example 45.** What is going on in the previous example?

To compare specific functions, let us consider initial values. Then,  $e^{ix}$  is the unique solution of  $y'' + y = 0$  which satisfies  $y(0) = 1$  and  $y'(0) = i$ . On the other hand, solving the IVP using  $y(x) = C \cos(x) + D \sin(x)$ , we get  $C = 1$  and  $D = i$ . This shows the fundamental identity

$$e^{ix} = \cos(x) + i \sin(x),$$

known as **Euler's identity**. ◇

**Remark 46.** Setting  $x = \pi$  in Euler's identity and rearranging, we get  $e^{i\pi} + 1 = 0$ , which combines the five most important mathematical constants in a single beautiful formula. ◇

**Definition 47.** Any complex number  $z \in \mathbb{C}$  can be written as  $z = x + iy$ , with  $x, y \in \mathbb{R}$ .  $x$  is called the **real part** and  $y$  the **imaginary part**. The complex **conjugate** of  $z$  is  $\bar{z} = x - iy$ .

From *abc*-formula: if  $z = x + iy$  is the root of a polynomial (with real coefficients), then so is  $\bar{z} = x - iy$ .

Its **absolute value** is  $r = |z| = \sqrt{x^2 + y^2}$ . Its **argument** (or **amplitude** or **phase**) is the angle  $\theta$  from the positive real axis to the vector  $(x, y)$  representing  $z$ .

In fact, this gives the **polar form**  $z = x + iy = r e^{i\theta}$ . [by Euler's identity!]

**Example 48.** Find the general solution of  $y'' + 4y' + 13y = 0$ .

**Solution.** The characteristic polynomial is  $r^2 + 4r + 13 = (r - (-2 + 3i))(r - (-2 - 3i))$ .

$y_1 = e^{(-2+3i)x} = e^{-2x} e^{3ix} = e^{-2x} (\cos(3x) + i \sin(3x))$ ,  $y_2 = e^{(-2-3i)x} = e^{-2x} e^{-3ix} = e^{-2x} (\cos(3x) - i \sin(3x))$

Note that  $\frac{1}{2}(y_1 + y_2) = e^{-2x} \cos(3x)$  and  $\frac{1}{2i}(y_1 - y_2) = e^{-2x} \sin(3x)$  are solutions as well. And they are real!

So, the general solution is  $A e^{-2x} \cos(3x) + B e^{-2x} \sin(3x)$ . This always works! ◇

**Theorem 49.** Consider, again, a homogeneous linear DE with constant coefficients.

- If  $r_0$  is a root of the characteristic polynomial and if  $k$  is its multiplicity, then  $e^{r_0 x}$ ,  $x e^{r_0 x}$ , ...,  $x^{k-1} e^{r_0 x}$  are solutions of the DE.
- If  $r_0 = a + bi$  is a complex root, then  $a - bi$  is another root, and we can write the corresponding solutions as  $e^{ax} \cos(bx)$  and  $e^{ax} \sin(bx)$ .

If the roots are repeated, we again have  $x^j e^{ax} \cos(bx)$  and  $x^j e^{ax} \sin(bx)$  as additional solutions.

**Example 50.** Find the general solution of  $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$ .

**Solution.** The characteristic polynomial factors as  $r^3(r^2 + 4r + 13)^2 = r^3(r - (-2 + 3i))^2(r - (-2 - 3i))^2$ .

Hence, the general solution is  $(A + Bx + Cx^2) + (D + Ex) e^{-2x} \cos(3x) + (F + Gx) e^{-2x} \sin(3x)$ . ◇

**Review.** complex numbers

◇

**Example 51.** Here is another way, to look at Euler's identity  $e^{ix} = \cos(x) + i \sin(x)$ .

For this identity to make sense, one needs to somehow characterize the exponential function on the left-hand side. Last time, we observed that both sides are the unique solution to the IVP  $y'' + y = 0$ ,  $y(0) = 1$ ,  $y'(0) = i$ . This time, we use the Taylor expansion  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ . Note that  $i^n$ , for  $n = 0, 1, 2, \dots$ , is  $1, i, -1, -i, 1, i, \dots$ . Hence, splitting the Taylor sum into even ( $n = 2m$ ) and odd ( $n = 2m + 1$ ) terms, we get

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \cos(x) + i \sin(x).$$

◇

**Example 52.** Euler's identity makes deriving many trig formulas easy!

For instance,  $\cos(x+y) = \operatorname{Re}(e^{i(x+y)}) = \operatorname{Re}(e^{ix}e^{iy}) = \operatorname{Re}((\cos(x) + i \sin(x))(\cos(y) + i \sin(y)))$   
 $= \cos(x)\cos(y) - \sin(x)\sin(y)$ .

◇

**Example 53.** Express the general solution of  $y''' - y'' + 4y' - 4y = 0$  using only real functions.

**Solution.** Characteristic polynomial  $r^3 - r^2 + 4r - 4$ . We spot the root 1. To find the other roots, we do polynomial division to get  $(r^3 - r^2 + 4r - 4)/(r - 1) = r^2 + 4$ . Hence, the characteristic polynomial has roots  $1, \pm 2i$ .

We therefore have the following solutions  $y_1 = \cos(2x)$ ,  $y_2 = \sin(2x)$ ,  $y_3 = e^x$ . The general solution is  $A \cos(2x) + B \sin(2x) + C e^x$ .

Our next goal is to think deeper about the final step, which allowed us to go from having three solutions to the general solution.

◇

## Independence of solutions

Given a homogeneous linear DE, we have learned that there *exist*  $n$  solutions  $y_1, y_2, \dots, y_n$ , such that the general solution is  $C_1 y_1 + \dots + C_n y_n$ . But if we find  $n$  solutions, how can we tell whether they give the general solution?

**Example 54.** Here are three other solutions of the previous example:  $u_1 = \cos(2x)$ ,  $u_2 = \sin(2x)$ ,  $u_3 = \cos(2x + 1)$ . However,  $c_1 u_1 + c_2 u_2 + c_3 u_3$  is not the general solution to the DE. Why?

Using the trig identity from Example 52, we see that  $\cos(1)u_1 - \sin(1)u_2 - u_3 = 0$ . This is a (linear) dependence relation between the solution functions and allows us to express one of them in terms of the other two. In other words, we really have only two solutions (the "really" means that we have only two independent solutions, a notion defined below) and are still missing a third one to get the general solution.

◇

**Definition 55.**  $n$  functions  $f_1, \dots, f_n$  are (linearly) dependent if there are coefficients  $c_1, \dots, c_n$ , not all zero, such that  $c_1 f_1 + \dots + c_n f_n = 0$ . Otherwise, they are called (linearly) **independent**.

**Theorem 56.** Suppose that  $y_1, y_2, \dots, y_n$  are solutions to a homogeneous linear DE of order  $n$ . Then the general solution is  $C_1 y_1 + \dots + C_n y_n$  if and only if  $y_1, y_2, \dots, y_n$  are independent.

**Example 57.** Are the functions  $u_1 = 3x^2 \sin^2(x)$ ,  $u_2 = 5x^2 \cos^2(x)$  and  $u_3 = x^2$  linearly independent?

**Solution.** Using  $\cos^2(x) + \sin^2(x) = 1$ , we find that  $\frac{1}{3}u_1 + \frac{1}{5}u_2 - u_3 = 0$ . In other words, the functions are linearly dependent.

◇

## Review. linear independence



Fix some  $a \in I$ . Note that  $y(x) = C_1 y_1(x) + \dots + C_n y_n(x)$  is the general solution of a HLDE<sup>8</sup> of order  $n$  if and only if we can solve for all initial values  $y(a) = b_0$ ,  $y'(a) = b_1$ , ...,  $y^{(n-1)}(a) = b_{n-1}$ .

Writing out these (linear) equations and expressing them in matrix form, we see that they are equivalent to finding  $(C_1, C_2, \dots, C_n)$  such that

$$\begin{pmatrix} y_1(a) & y_2(a) & \cdots & y_n(a) \\ y_1'(a) & y_2'(a) & \cdots & y_n'(a) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(a) & y_2^{(n-1)}(a) & \cdots & y_n^{(n-1)}(a) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}.$$

Linear Algebra<sup>9</sup> tells us that this system of linear equations can be solved for all values of the  $b_j$  if and only if the determinant of the matrix on the LHS is not zero. This determinant is the Wronskian  $W(a)$  of  $y_1, \dots, y_n$ .

**Definition 58.** The **Wronskian** of the  $n$  functions  $f_1, \dots, f_n$  is the  $n \times n$  determinant

$$W(x) := \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Note that, for linearly dependent functions,  $W(x) = 0$  for all  $x$ . Why?!

**Theorem 59.** Solutions  $y_1, \dots, y_n$  of a homogeneous linear DE of order  $n$  are linearly independent if and only if  $W(x) \neq 0$  for some  $x \in I$ . [in which case  $W(x) \neq 0$  for all  $x \in I$ ]

**Example 60.**  $y'' + 4y' + 4y = 0$  has solutions  $y_1 = e^{-2x}$ ,  $y_2 = xe^{-2x}$ .

The Wronskian is  $\begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{vmatrix} = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}[1-2x+2x] = e^{-4x} \neq 0$ .

Hence,  $y_1, y_2$  are independent and the general solution is  $y(x) = Ay_1(x) + By_2(x)$ . ◇

**Example 61.**  $y''' = 0$  has solutions  $y_1 = 3$ ,  $y_2 = 1 - 2x^2$ ,  $y_3 = 5x^2$ . Are these independent?

**Solution.** No, because  $y_1 - 3y_2 - \frac{6}{5}y_3 = 0$ .

**Solution.** No, because  $W(0) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 10 \end{vmatrix} = 0$ . [evaluating the Wronskian at 0 makes our computation easy!]

What about the solutions  $y_1 = 3$ ,  $y_2 = 1 - 2x^2$ ,  $y_3 = 5x$ . Are they independent?

**Solution.** Yes, because  $W(0) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & -4 & 0 \end{vmatrix} = 60$ . ◇

**Remark.** JFF<sup>10</sup>. The Riemann zeta function is defined by the sum  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ , which converges if  $\text{Re } s > 1$ . For other complex values of  $s$ , there is a unique way to “analytically continue” this function. It is then “easy” to see that  $\zeta(-2) = 0$ ,  $\zeta(-4) = 0$ , .... The **Riemann hypothesis** claims that all other zeroes of  $\zeta(s)$  lie on the line  $\text{Re}(s) = \frac{1}{2}$ . A proof of this conjecture (checked for the first 10,000,000,000 zeroes) is worth<sup>11</sup> \$1,000,000. ◇

8. Writing the DE as  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = 0$ , we need the coefficients  $p_j(x)$  to be at least continuous on the interval  $I$ .

9. Don't worry if you are not familiar with this, as we will go over basics of Linear Algebra when we really need it. However, it may be a good idea to start reading up on matrices and vectors because we will be brief.

10. Just for fun.

11. <http://www.claymath.org/millennium-problems/riemann-hypothesis>



## Review

- Basic understanding
  - DEs and IVPs
  - existence and uniqueness
  - visualization of first-order DEs via slope fields
- Basic modeling
  - population models
  - modeling simple motions
  - mixing problems
- Solving techniques
  - linear DEs with constant coefficients
  - separation of variables ( $y' = f(x)g(y)$ )
  - linear first-order equations (integrating factor)
  - common substitutions (e.g.  $y' = f(y/x)$ )

**Example 62.** What can we say about existence and uniqueness of the initial value problem  $\frac{dy}{dx} = \ln(x^2 + y^2) - \frac{\cos(y^3)}{x}$ ,  $y(1) = 0$ ?

**Solution.** Write as  $y' = f(x, y)$  with  $f(x, y) = \ln(x^2 + y^2) - \frac{\cos(y^3)}{x}$ . Then  $\frac{\partial}{\partial y} f(x, y) = \frac{2y}{x^2 + y^2} + \frac{3y^2 \sin(y^3)}{x}$ . We observe that both  $f(x, y)$  and  $\frac{\partial}{\partial y} f(x, y)$  are continuous for all  $(x, y)$  with  $x \neq 0$ . In particular,  $f(x, y)$  and  $\frac{\partial}{\partial y} f(x, y)$  are continuous around  $(1, 0)$ . Consequently, the IVP has a solution and it is unique.  $\diamond$

**Example 63.**  $x^2(2y^3 - y)\frac{dy}{dx} = xy^5 - y^5$

**Solution.** This equation is separable! (Note that  $xy^5 - y^5 = y^5(x - 1)$ .)  $\diamond$

**Example 64.**  $xy' = 4x^4 - (2x - 3)y$

**Solution.** This equation is linear! Write as  $y' + \frac{2x-3}{x}y = 4x^3$ , determine integrating factor, ...  $\diamond$

**Example 65.**  $\frac{dy}{dx} = \frac{xy + \sec^2(x)\cos(y) + x\cos(y) + y\sec^2(x)}{1 - \sin(y)}$

**Solution.** This equation is separable!  $\frac{dy}{dx} = \frac{(x + \sec^2(x))(y + \cos(y))}{1 - \sin(y)}$   $\diamond$



## Inhomogeneous linear DEs

Recall that a **linear DE** is one of the form  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = f(x)$ . Writing<sup>12</sup>  $D = \frac{d}{dx}$  and setting  $L := D^n + p_{n-1}(x)D^{n-1} + \dots + p_1(x)D + p_0(x)$ , this DE takes the concise form  $L \cdot y = f(x)$ . [ $L$  is a **linear differential operator**.]

- Note that  $L \cdot (y_1 + y_2) = L \cdot y_1 + L \cdot y_2$ .  
In particular, if  $L \cdot y_1 = 0$  and  $L \cdot y_2 = 0$ , then  $L \cdot (y_1 + y_2) = 0$ . Superposition!  
More generally,  $L \cdot (C_1 y_1 + C_2 y_2) = C_1 L \cdot y_1 + C_2 L \cdot y_2$ .
- Let  $y_p$  be a particular solution to  $L \cdot y = f(x)$ . Let  $C_1 y_1 + \dots + C_n y_n$  be the general solution of  $L \cdot y = 0$ . Then  $y_p + C_1 y_1 + \dots + C_n y_n$  is the **general solution of  $L \cdot y = f(x)$** .  
Hence, solving an inhomogeneous linear DE reduces to two simpler problems!

**Example 66.** Find the general solution of  $y'' + 4y = 12x$ .

*Hint:*  $3x$  is a solution.

**Solution.** Here,  $L = D^2 + 4$ . We already know one solution  $y_p = 3x$ .

Solving  $L \cdot y = 0$  gives  $y_1 = \cos(2x)$  and  $y_2 = \sin(2x)$ .

[Make sure this is easy for you!]

Therefore, the general solution is  $y_p + C_1 y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$ .

How to find the particular solution ourselves? Apply  $D^2$  to the DE! We get  $D^2(D^2 + 4) \cdot y = 0$ , which is a homogeneous linear DE! Its general solution is  $C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$ . In particular,  $y_p$  is of this form for some choice of  $C_1, \dots, C_4$ .

In fact, it simplifies our life to note that there has to be a particular solution of the simpler form  $y_p = C_1 + C_2 x$ . Why?!

Now, it only remains to find appropriate values  $C_1, C_2$  such that  $y_p'' + 4y_p = 12x$ .  $y_p'' + 4y_p = 4C_1 + 4C_2 x = 12x$  and we conclude  $C_1 = 0$  and  $C_2 = 3$ . We found  $y_p = 3x$ , as used above. ◇

**Example 67.** Find the general solution of  $y'' + 4y' + 4y = e^x$ .

**Solution.** Here,  $L = D^2 + 4D + 4 = (D + 2)^2$ . The general solution of  $L \cdot y = 0$  is  $(C_1 + C_2 x)e^{-2x}$ .

Note that  $(D - 1) \cdot e^x = 0$ . Hence, we apply  $(D - 1)$  to the DE to get  $(D - 1)(D + 2)^2 \cdot y = 0$ . This homogeneous linear DE has general solution  $(C_1 + C_2 x)e^{-2x} + C_3 e^x$ . We conclude that the original DE must have a particular solution  $y_p = C_3 e^x$ . To determine the value of  $C_3$ , we plug into the DE:  $y_p'' + 4y_p' + 4y_p = 9C_3 e^x = e^x$ . Hence,  $C_3 = 1/9$ . Finally, the general solution is  $(C_1 + C_2 x)e^{-2x} + \frac{1}{9}e^x$ . ◇

This method gives a **recipe for solving nonhomogeneous linear DEs with constant coefficients**.

It works whenever the right-hand side  $f(x)$  is the solution of some homogeneous linear DE.

**Theorem 68.** To find a particular solution  $y_p$  of  $L \cdot y = f(x)$ .

- Let  $r_1, \dots, r_n$  be the (old) roots of the char poly of  $L \cdot y = 0$ .
- Let  $s_1, \dots, s_m$  be the (new) roots of the char poly of  $L_{\text{rhs}} \cdot f = 0$ , the HLDE (with constant coefficients) which  $f(x)$  solves. (This is not possible for all  $f(x)$ .)
- It follows that  $y_p$  solves  $L_{\text{rhs}} L \cdot y = 0$ . Its char poly has roots  $r_1, \dots, r_n, s_1, \dots, s_m$ . Let  $v_1, \dots, v_m$  be the “new” solutions (i.e. not solutions of the “old”  $L \cdot y = 0$ ). Now, we can find (unique) constants  $C_i$  such that  $y_p = C_1 v_1 + \dots + C_m v_m$ .

**Example 69.** Find the general solution of  $y'' + 4y' + 4y = 7e^{-2x}$ .

**Solution.** “Old” roots  $-2, -2$ . “New” roots  $-2$ . Hence, there has to be a particular solution of the form  $y_p = Cx^2 e^{-2x}$ . To find the value of  $C$ , we need to plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$ .  $y_p'' = C(4x^2 - 8x + 2)e^{-2x}$ . Hence,  $y_p'' + 4y_p' + 4y_p = 2C e^{-2x} \stackrel{!}{=} 7e^{-2x}$ .  $C = 7/2$ .

Since  $y_p = \frac{7}{2}x^2 e^{-2x}$ , the general solution is  $(C_1 + C_2 x + \frac{7}{2}x^2)e^{-2x}$ . ◇

<sup>12</sup> As in the proof of Theorem 42.

**Review.** solving nonhomogeneous linear DEs with constant coefficients

◇

**Example 70.** Find a particular solution of  $y'' + 4y' + 4y = 7e^{-2x}$ .

**Solution.** Again,  $L = D^2 + 4D + 4 = (D + 2)^2$ .

“Old” roots  $-2, -2$ . “New” roots  $-2$ . Hence, there has to be a particular solution of the form  $y_p = Cx^2e^{-2x}$ . To find the value of  $C$ , we need to plug into the DE.

$$y'_p = C(-2x^2 + 2x)e^{-2x}$$

$$y''_p = C(4x^2 - 8x + 2)e^{-2x}$$

$$y''_p + 4y'_p + 4y_p = 2Ce^{-2x} \stackrel{!}{=} 7e^{-2x}. \quad C = 7/2.$$

Hence,  $y_p = \frac{7}{2}x^2e^{-2x}$ . (Last time, we didn't finish the computation.)

◇

**Example 71.** Find a particular solution of  $y'' + 4y' + 4y = x \cos(x)$ .

**Solution.** “Old” roots  $-2, -2$ . “New” roots  $\pm i, \pm i$ . Hence, there has to be a particular solution of the form  $y_p = (C_1 + C_2x)\cos(x) + (C_3 + C_4x)\sin(x)$ . To find the value of the  $C_j$ 's, we need to plug into the DE.

$$y'_p = (C_2 + C_3 + C_4x)\cos(x) + (C_4 - C_1 - C_2x)\sin(x)$$

$$y''_p = (2C_4 - C_1 - C_2x)\cos(x) + (-2C_2 - C_3 - C_4x)\sin(x)$$

$$y''_p + 4y'_p + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$$

$$+ (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \stackrel{!}{=} x \cos(x).$$

Equating the coefficients of  $\cos(x)$ ,  $x \cos(x)$ ,  $\sin(x)$ ,  $x \sin(x)$ , we get the equations  $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$ ,  $3C_2 + 4C_4 = 1$ ,  $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$ ,  $-4C_2 + 3C_4 = 0$ .

Solving, we find  $C_1 = -\frac{4}{125}$ ,  $C_2 = \frac{3}{25}$ ,  $C_3 = -\frac{22}{125}$ ,  $C_4 = \frac{4}{25}$ . [Make sure you know how to do this tedious step.]

Hence,  $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$ .

◇

**Example 72.** Find a particular solution of  $y'' + 4y' + 4y = 5e^{-2x} - 3x \cos(x)$ .

**Solution.** Instead of starting all over, recall that we already found  $y_\Delta$  in Example 70 such that  $Ly_\Delta = 7e^{-2x}$ .

Also, from Example 71 we have  $y_\diamond$  such that  $Ly_\diamond = x \cos(x)$ .

By linearity, it follows that  $L\left(\frac{5}{7}y_\Delta - 3y_\diamond\right) = \frac{5}{7}Ly_\Delta - 3Ly_\diamond = 5e^{-2x} - 3x \cos(x)$ .

Hence,  $y_p = \frac{5}{7}y_\Delta - 3y_\diamond = \frac{5}{2}x^2e^{-2x} - 3\left[\left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)\right]$ .

◇

**Example 73.** Find a particular solution of  $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x \sin(x)$ .

**Solution.** “Old” roots  $-2, -2$ . “New” roots  $3 \pm 2i, \pm i, \pm i$ .

Hence, there has to be a particular solution of the form

$$y_p = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + (C_3 + C_4x)\cos(x) + (C_5 + C_6x)\sin(x).$$

To find the values of  $C_1, \dots, C_6$ , we plug into the DE. But this final step is so boring that we stop here.

Computers (currently?) cannot afford to be as selective; mine obediently calculated:

$$y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$$

◇

**Theorem 74.** The linear DE  $Ly = y'' + P(x)y' + Q(x)y = f(x)$  has particular solution

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx,$$

where  $y_1, y_2$  are independent solutions of  $Ly = 0$  and  $W = y_1y_2' - y_1'y_2$  is the Wronskian of  $y_1, y_2$ .

[Note that considering all possible constants of integration actually gives the general solution of  $Ly = f(x)$ .]

**Proof.** Let us look for  $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ . This “ansatz” is called [variation of constants/parameters](#).

Then  $y_p' = \underbrace{u_1'y_1 + u_2'y_2}_{=0 \text{ (or so we wish)}} + u_1y_1' + u_2y_2'$  and  $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$ .

$$\begin{aligned} Ly_p &= u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) \\ &= u_1'y_1' + u_2'y_2' + u_1Ly_1 + u_2Ly_2 = u_1'y_1' + u_2'y_2' \end{aligned}$$

So, in order for  $y_p = u_1y_1 + u_2y_2$  to solve  $Ly = f(x)$ , we need

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= f(x). \end{aligned}$$

These are linear equations in  $u_1'$  and  $u_2'$ . Solving gives  $u_1' = \frac{-y_2 f(x)}{y_1y_2' - y_1'y_2}$  and  $u_2' = \frac{y_1 f(x)}{y_1y_2' - y_1'y_2}$ , and it only remains to integrate.  $\square$

**Example 75.** Find a particular solution of  $y'' - 2y' + y = \frac{e^x}{x}$ .

**Solution.** Here,  $y_1 = e^x$ ,  $y_2 = xe^x$ . We calculate  $W(x) = e^{2x}$ .

$y_p = -e^x \int \frac{1}{x} dx + xe^x \int \frac{1}{x} dx = xe^x [\ln|x| - 1]$ . (Note that, with integration constants, we get  $-e^x(x + C_1) + xe^x(\ln|x| + C_2)$ , which is the general solution. So any constants suffice to give us a particular solution.)  $\diamond$

**Example 76.** Solve  $Ly = x^2y'' - 4xy' + 6y = x^3$ . Given:  $y_1 = x^2$  and  $y_2 = x^3$  solve  $Ly = 0$ .

**Solution.** First,  $W(x) = x^4$ . Put DE in the form  $y'' - 4x^{-1}y' + 6x^{-2}y = x$ .

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx = -x^2 \int \frac{1}{x} dx + x^3 \int \frac{1}{x} dx = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is  $C_1x^2 + (C_2 + \ln|x|)x^3$ .  $\diamond$

**Remark 77.** Just for fun (and understanding and context), let us revisit our method of solving first-order linear DEs  $Ly = y' + P(x)y = f(x)$ . [Variation of constants also extends to higher-order DEs.]

Note that  $Ly = 0$  has solution  $y_1 = \exp(-\int P(x)dx)$ , which is nothing but the inverse of the integrating factor!

Using the integrating factor, we arrive at  $y_p(x) = y_1(x) \int \frac{f(x)}{y_1(x)} dx$  which is the analog of Theorem 74.  $\diamond$

## Application: motion of a pendulum

The motion of an (ideal) pendulum is described by  $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ , where  $\theta$  is the angular displacement and  $L$  is the length of the pendulum (and, as usual,  $g$  is acceleration due to gravity).

**Proof.** We assume the string to be massless, and let  $m$  be the swinging mass.

Let  $s$  and  $h$  be as in the picture on the right.

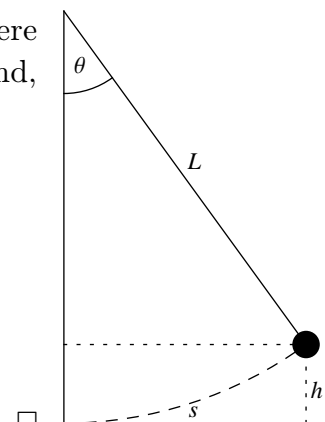
Velocity (more accurately, speed) of mass:  $v = \frac{ds}{dt} = L \frac{d\theta}{dt}$

Kinetic energy:  $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2$

Potential energy:  $V = mgh = mgL(1 - \cos\theta)$  (weight  $mg$  times height  $h$ )

Conservation of energy:  $T + V = \text{const}$

Take time derivative:  $\frac{1}{2}mL^2 2 \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} + mgL \sin\theta \frac{d\theta}{dt} = 0$ . Finally, cancel terms.  $\square$



**Review.** Let  $\theta$  be angular displacement of a pendulum on a string of length  $L$ . Then its motion is described by  $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$ .  $\sin\theta \approx \theta$  for small  $\theta$  so, approximately<sup>13</sup>, we get  $\theta'' + \frac{g}{L}\theta = 0$ . (This time, we used Newton's second law and considered the tangential component of the gravitational force to derive the equation of motion.  $F = -\sin\theta \cdot mg$  and  $F = ma = mL\theta''$ .)  $\diamond$

**Example 78.** The motion of a mass attached to a spring  $mx'' + kx = 0$ .

Here,  $x$  is displacement from equilibrium,  $k$  spring constant (for instance, N/m),  $m$  mass. Hooke's law:  $F = -kx$ . Newton:  $F = ma = mx''$ .  $\diamond$

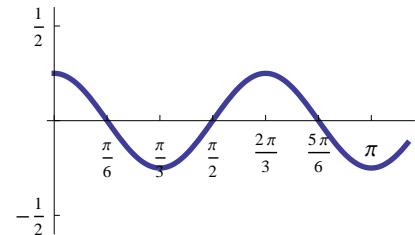
**Example 79.** Solve the IVP  $\theta'' + 9\theta = 0$ ,  $\theta(0) = 1/4$  (about  $14.3^\circ$ ),  $\theta'(0) = 0$ .

**Solution.** The roots of the characteristic polynomial are  $\pm 3i$ .  
Hence,  $\theta(t) = A \cos(3t) + B \sin(3t)$ .  $\theta(0) = A = 1/4$ .  $\theta'(0) = 3B = 0$ .  
Therefore, the solution is  $\theta(t) = 1/4 \cos(3t)$

Amplitude:  $1/4$

Period:  $T = \frac{2\pi}{3}$

(Circular) frequency:  $3$



$\diamond$

**Example 80.** Solve  $\theta'' + 9\theta = 0$ ,  $\theta(0) = 1/4$ ,  $\theta'(0) = -3/2$  ("initial kick"). What is the amplitude?

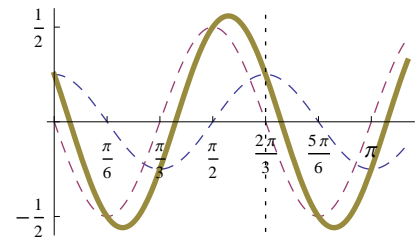
**Solution.** This time,  $\theta(0) = A = 1/4$ .  $\theta'(0) = 3B = -3/2$ .

Hence,  $\theta(t) = \frac{1}{4} \cos(3t) - \frac{1}{2} \sin(3t) = \frac{\sqrt{5}}{4} \cos(3t - \alpha)$ .

The last equality follows because  $(\frac{1}{4}, -\frac{1}{2}) = r(\cos \alpha, \sin \alpha)$  with  $r = \frac{\sqrt{5}}{4}$  and  $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$ . (see next Example and Review)

The amplitude is  $\frac{\sqrt{5}}{4} \approx 0.559$ .

Phase angle:  $\alpha$  (or time lag  $\alpha/3$ )



$\diamond$

**Example 81.**  $A \cos(\omega t) + B \sin(\omega t) = r \cos(\omega t - \alpha)$  with  $(r, \alpha)$  the polar coordinates for  $(A, B)$

If you like trig identities:  $A \cos(\omega t) + B \sin(\omega t) = r(\cos(\alpha)\cos(\omega t) + \sin(\alpha)\sin(\omega t)) = r \cos(\omega t - \alpha)$ .

If you like DEs: both sides solve  $x'' + \omega^2 x = 0$ . The LHS has initial values  $y(0) = A$  and  $y'(0) = \omega B$ , the RHS has  $y(0) = r \cos(\alpha)$  and  $y'(0) = r\omega \sin(\alpha)$ . Hence, the two are equal if  $A = r \cos(\alpha)$  and  $B = r \sin(\alpha)$ .  $\diamond$

**Review 82.** How to calculate the polar coordinates  $(r, \alpha)$  for  $(A, B)$ ?

We need to find  $r \geq 0$  and  $\alpha \in [0, 2\pi)$  such that  $(A, B) = r(\cos \alpha, \sin \alpha)$ . Hence,  $r = \sqrt{A^2 + B^2}$  and  $\alpha$  is determined by  $\cos(\alpha) = \frac{A}{r}$  and  $\sin(\alpha) = \frac{B}{r}$ . In particular,  $\tan(\alpha) = \frac{B}{A}$  and, if careful, we can compute  $\alpha$  using  $\tan^{-1}$  as

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) + \begin{cases} 0, & \text{if } (A, B) \text{ in first quadrant,} \\ 2\pi, & \text{if } (A, B) \text{ in fourth quadrant,} \\ \pi, & \text{otherwise.} \end{cases}$$

$\diamond$

13. At least for short times and small angles. For instance, for  $\theta = 15^\circ$  the error  $\theta - \sin\theta$  is about 1%.

## The qualitative effects of damping

Let us consider  $x'' + dx' + cx = 0$  with  $c > 0$  and  $d \geq 0$ . The term  $dx'$  models damping (e.g. friction, air resistance) proportional to the velocity  $x'$ .

The characteristic equation  $r^2 + dr + c = 0$  has roots  $\frac{-d \pm \sqrt{d^2 - 4c}}{2}$ . The nature of the solutions depends on whether the **discriminant**  $\Delta = d^2 - 4c$  is positive, negative, or zero.

**Undamped.**  $d = 0$ . In that case,  $\Delta < 0$ . Two complex roots  $\pm i\omega$  with  $\omega = \sqrt{c}$ .

Solutions:  $c_1 \cos(\omega t) + c_2 \sin(\omega t) = r \cos(\omega t - \alpha)$  where  $(c_1, c_2) = r(\cos \alpha, \sin \alpha)$

Oscillations with frequency  $\omega = \sqrt{c}$ , period  $\frac{2\pi}{\sqrt{c}}$ , time lag  $\frac{\alpha}{\sqrt{c}}$

**Underdamped.**  $d > 0$ ,  $\Delta < 0$ . Two complex roots  $-\rho \pm i\omega$  with  $-\rho = -d/2 < 0$ .

Solutions:  $e^{-\rho t}[c_1 \cos(\omega t) + c_2 \sin(\omega t)] = e^{-\rho t}[r \cos(\omega t - \alpha)]$  ( $\rightarrow 0$  as  $t \rightarrow \infty$ )

Oscillations with amplitude going to zero

**Critically damped.**  $d > 0$ ,  $\Delta = 0$ . One (double) real root  $-\rho < 0$ .

Solutions:  $(c_1 + c_2 t)e^{-\rho t}$  ( $\rightarrow 0$  as  $t \rightarrow \infty$ )

No oscillations (at most one crossing of  $t$ -axis; why?!)

**Overdamped.**  $d > 0$ ,  $\Delta > 0$ . Two real roots  $-\rho_1, -\rho_2 < 0$ .

[negative because  $c, d > 0$ ]

Solutions:  $c_1 e^{-\rho_1 t} + c_2 e^{-\rho_2 t}$  ( $\rightarrow 0$  as  $t \rightarrow \infty$ )

No oscillations (at most one crossing of  $t$ -axis)

## Adding external forces and the phenomenon of resonance

**Example 83.** A car is going at constant speed  $v$  on a washboard road surface (“bumpy road”) with height profile  $y(s) = a \cos\left(\frac{2\pi s}{L}\right)$ . Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

**Solution.** With  $x$  as in the sketch, the spring is stretched by  $x - y$ . Hence, by Hooke’s and Newton’s laws, its motion is described by  $mx'' = -k(x - y)$ .

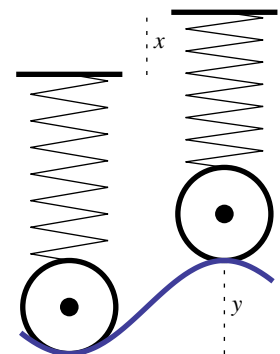
At constant speed,  $s = vt$  and we obtain the DE  $mx'' + kx = ky = ka \cos\left(\frac{2\pi vt}{L}\right)$ , which is inhomogeneous linear with constant coefficients. Let’s solve it.

“Old” roots  $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$ .  $\omega_0 = \sqrt{\frac{k}{m}}$  is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces).

“New” roots  $i\frac{2\pi v}{L} = \pm i\omega$ .  $\omega = \frac{2\pi v}{L}$  is the **external frequency**.

**Case 1:  $\omega \neq \omega_0$ .** Then a particular solution is  $x_p = b_1 \cos(\omega t) + b_2 \sin(\omega t) = A \cos(\omega t - \alpha)$  for unique values of  $b_1, b_2$  (which we do not compute here). The general solution is of the form  $x = x_p + C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ .

**Case 2:  $\omega = \omega_0$ .** Then a particular solution is  $x_p = t[b_1 \cos(\omega t) + b_2 \sin(\omega t)] = At \cos(\omega t - \alpha)$  for unique values of  $b_1, b_2$  (which we do not compute). Note that the amplitude in  $x_p$  increases without bound; the same is true for the general solution  $x = x_p + C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ . This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.



The first “car” is assumed to be in equilibrium.

◇

Note that resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

**Example 84.** Consider  $x'' + 9x = 10 \cos(2vt)$ . For what value of  $v$  does resonance occur?

**Solution.** The natural frequency is 3. The external frequency is  $2v$ . Hence, resonance occurs when  $v = \frac{3}{2}$ . ◇

## External forces plus damping

**Example 85.** Find the general solution of  $2x'' + 2x' + x = 10 \sin(t)$ .

**Solution.** “Old” roots  $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$ . So the system without external force is underdamped. [Why?!] After a routine calculation,  $x_p = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t - \alpha))$  with  $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$ . Here, we used that  $(-4, -2) = \sqrt{20}(\cos \alpha, \sin \alpha)$ .

Hence, the general solution is  $x(t) = \underbrace{\sqrt{20} \cos(t - \alpha)}_{x_{sp}} + \underbrace{e^{-t/2}(c_1 \cos(t/2) + c_2 \sin(t/2))}_{x_{tr} \rightarrow 0 \text{ as } t \rightarrow \infty}$ .

Observe how  $x = x_{tr} + x_{sp}$  splits into **transient** motion  $x_{tr}$  and **steady periodic** oscillations  $x_{sp}$ . ◇

**Example 86.** Find the steady periodic solution to  $x'' + 2x' + 5x = \cos(\omega t)$ . What is the amplitude of the steady periodic oscillations? For which  $\omega$  is the amplitude maximal?

**Solution.** “Old” roots  $-1 \pm 2i$ . [Not really needed, because positive damping prevents duplication; can you see it?]

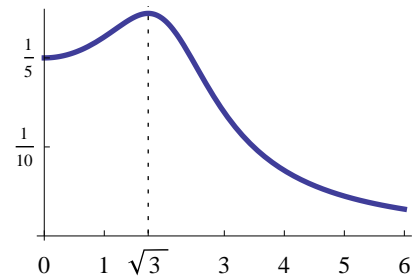
Hence,  $x_{sp} = A_1 \cos(\omega t) + A_2 \sin(\omega t)$  and to find  $A_1, A_2$  we need to plug into the DE.

Doing so (we did!), we find  $A_1 = \frac{5 - \omega^2}{(5 - \omega^2)^2 + 4\omega^2}$ ,  $A_2 = \frac{2\omega}{(5 - \omega^2)^2 + 4\omega^2}$ .

Consequently, the amplitude of  $x_{sp}$  is  $A_{sp} = \sqrt{A_1^2 + A_2^2} = \frac{1}{\sqrt{(5 - \omega^2)^2 + 4\omega^2}}$ .

The function  $A_{sp}(\omega)$  is sketched to the right. It has a maximum at  $\omega = \sqrt{3}$  at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for  $\omega = \sqrt{3}$ .

[For comparison, without damping, (pure) resonance occurs for  $\omega = \sqrt{5}$ .]



◇

## Systems of differential equations

**Example 87.** Consider two springs attached to each other as in Figure 4.1.1.

Write  $x_1(t)$  for the displacement of mass  $m_1$  from equilibrium and, likewise,  $x_2(t)$  for the mass  $m_2$ . Note that the first spring is stretched by  $x_1$  whereas the second spring is stretched by  $x_2 - x_1$ . Applying Hooke's law and Newton's second law to each mass, while assuming the other one to be stationary, we find that

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2 (x_2 - x_1), \\ m_2 x_2'' &= -k_2 (x_2 - x_1). \end{aligned}$$

This is a **system of differential equations**. This particular one is linear and second-order.

Of course, now one can again introduce damping, external forces, etc. ◇

**Fact.** Any DE (or system) can be transformed into a **first-order system** of DEs! ◇

**Example 88.** Write  $y''' + a(x)y'' + b(x)y' + c(x)y = f(x)$  as a first-order system.

**Solution.** Introduce  $y_1 = y$ ,  $y_2 = y'$ ,  $y_3 = y''$ . Then

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= -c(x)y_1 - b(x)y_2 - a(x)y_3 + f(x). \end{aligned}$$

This system is equivalent to the original DE in that  $y$  solves the original DE if and only if  $(y_1, y_2, y_3) = (y, y', y'')$  solves the system of DEs.

By the way, the above system can be expressed in matrix-vector notation as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c(x) & -b(x) & -a(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}, \quad \text{or, for short, } \mathbf{y}' = A(x)\mathbf{y} + F(x),$$

where, in the final expression,  $A(x)$  is the  $3 \times 3$  matrix and  $F(x)$  the vector. This is not just cosmetics but understanding matrices will allow us to use similar techniques as before; in many ways, we will be able to treat  $A$  just like a number. ◇



**Review.** systems of differential equations; express DEs as first-order systems ◇

**Example 89.** Express the non-linear DE  $x'' = x^3 + (x')^3$  as a first-order system<sup>14</sup>.

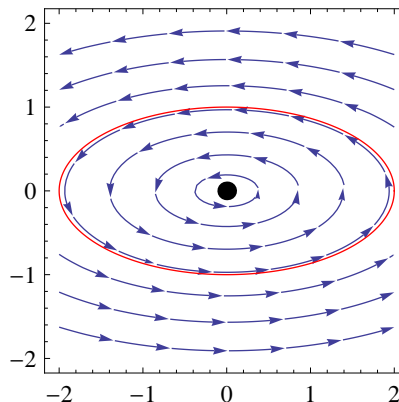
**Solution.** Introduce  $x_1 = x$ ,  $x_2 = x'$  to obtain the system  $x'_1 = x_2$ ,  $x'_2 = x_1^3 + x_2^3$ . ◇

**Example 90.** Solve the system  $x' = -2y$ ,  $y' = \frac{1}{2}x$ .

**Solution.** Observe that  $x'' = -2y' = -x$ . It follows that  $x(t) = B_1 \cos(t) + B_2 \sin(t) = A \cos(t - \alpha)$  where  $(B_1, B_2) = A(\cos(\alpha), \sin(\alpha))$ . Consequently,  $y = -\frac{1}{2}x' = \frac{1}{2}A \sin(t - \alpha)$ .

Since  $\cos^2 + \sin^2 = 1$ , each solution  $(x(t), y(t)) = \left(A \cos(t - \alpha), \frac{1}{2}A \sin(t - \alpha)\right)$  satisfies  $\frac{x^2}{A^2} + \frac{y^2}{(A/2)^2} = 1$ . This describes an ellipse! Several such curves are depicted in the **phase plane portrait** on the right which plots points  $(x, y)$  (also included are arrows to indicate evolution in time).

Note that we can use  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}x}{-2y} = -\frac{1}{4} \frac{x}{y}$  to sketch the phase portrait (just like for slope fields) without finding the solutions first. Do you also see how to get the directions? ◇



**Example 91.** Solve  $x' = -2y$ ,  $y' = \frac{1}{2}x$ ,  $x(0) = 2$ ,  $y(0) = 0$ .

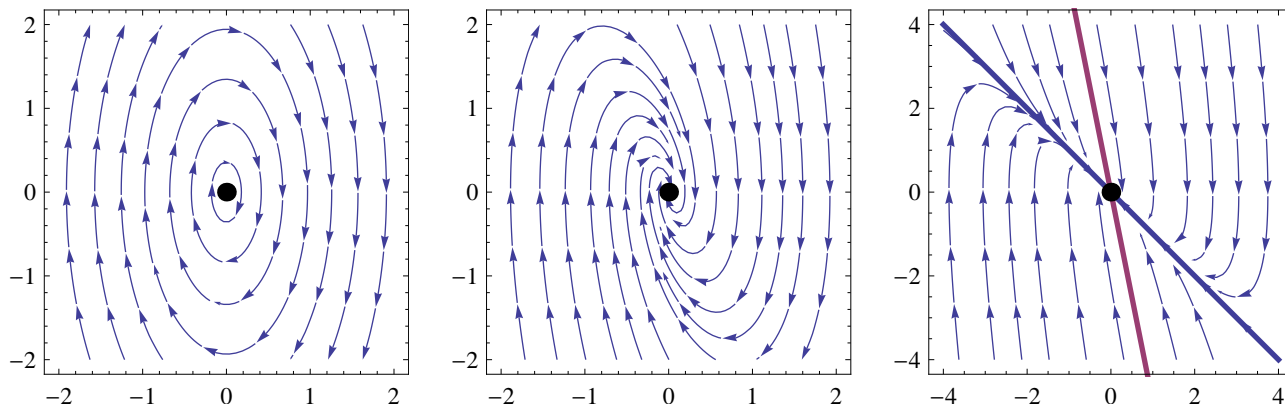
**Solution.** From before,  $x(t) = B_1 \cos(t) + B_2 \sin(t)$  and  $y(t) = -\frac{1}{2}x'(t) = \frac{1}{2}B_1 \sin(t) - \frac{1}{2}B_2 \cos(t)$ . Hence,  $x(0) = B_1 = 2$  and  $y(0) = -B_2/2 = 0$ . The solution  $x(t) = 2\cos(t)$ ,  $y(t) = \sin(t)$  is the red curve in the phase portrait. ◇

**Example 92.** Let  $x'' + dx' + cx = 0$  describe the motion of a mass on a spring. Besides  $x$  (position), introduce  $y = x'$  (velocity) to write it as the system  $x' = y$ ,  $y' = -cx - dy$ .

**undamped.**  $x'' + 4x = 0$  (roots  $\pm 2i$ ) translates into  $x' = y$ ,  $y' = -4x$ . The solutions are  $x = C \cos(2t - \alpha)$  and  $y = x' = -2C \sin(2t - \alpha)$ . As in the previous example, the points  $(x(t), y(t))$  lie on an ellipse. Look at a trajectory of the phase portrait and describe the physical meaning of its path.

**underdamped.**  $x'' + 2x' + 5x = 0$  (roots  $-1 \pm 2i$ ) translates into  $x' = y$ ,  $y' = -5x - 2y$ . The solutions are  $x = e^{-t}(B_1 \cos(2t) + B_2 \sin(2t))$  and  $y = x' = e^{-t}((2B_2 - B_1)\cos(2t) - (2B_1 + B_2)\sin(2t))$ . The points  $(x(t), y(t))$  spiral towards the origin.

**overdamped.**  $x'' + 6x' + 5x = 0$  (roots  $-1, -5$ ) translates into  $x' = y$ ,  $y' = -5x - 6y$ . The solutions are  $x = B_1 e^{-t} + B_2 e^{-5t}$  and  $y = x' = -B_1 e^{-t} - 5B_2 e^{-5t}$ . The points  $(x(t), y(t))$  approach the origin. The trajectories are simple for the special solutions  $(x, y) = (e^{-t}, -e^{-t}) = e^{-t}(1, -1)$  and  $(x, y) = (e^{-5t}, -5e^{-5t}) = e^{-5t}(1, -5)$ ; in both cases, these are just lines. [Other solutions are a combination of these two, but the trajectories move much faster in direction of the second line. Do you see why?!] ◇



<sup>14</sup> Such translations are useful for practical purposes. For instance, it means that one only needs to develop numerical algorithms for first-order systems.

## A crash course in linear algebra

**Example 93.** A typical  $2 \times 3$  matrix is  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ . It is composed of column vectors like  $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$  and row vectors like  $(1 \ 2 \ 3)$ .

Matrices (and vectors) of the same dimensions can be added and multiplied by a scalar. For instance,  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 5 \\ 6 & 8 & 5 \end{pmatrix}$  or  $3 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$ .  $\diamond$

**Remark 94.** More generally, a **vector space** is an abstraction of a collection of objects that can be added and scaled: numbers, lists of numbers (like the above row and column vectors), arrays of numbers (like the above matrices), arrows, functions, polynomials, differential operators, solutions to homogeneous linear differential equations, ...  $\diamond$

**Example 95.** The **transpose**  $A^T$  of a matrix  $A$  is the matrix obtained by interchanging the roles of rows and columns. For instance,  $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ .  $\diamond$

**Example 96.** Matrices of appropriate dimensions can also be **multiplied**.

This is based on the multiplication  $(x_1 \ x_2 \ x_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$  of row and column vectors.

For instance,  $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}$ . In general, we can multiply a  $m \times n$  matrix  $A$  with a  $n \times r$  matrix  $B$  to get a  $m \times r$  matrix  $AB$ . Its entry in row  $i$  and column  $j$  is defined to be  $(AB)_{ij} = (\text{row } i \text{ of } A) \begin{pmatrix} \text{column } j \\ \text{of } B \end{pmatrix}$ .

A good way to think about the multiplication  $A\mathbf{x}$  is that the resulting vector is a linear combination of the columns of  $A$  with coefficients from  $\mathbf{x}$ . Similarly, we can think of  $\mathbf{x}^T A$  as a combination of the rows of  $A$ .

Some nice properties of matrix multiplication are:

- There is a  $n \times n$  identity matrix  $I$  (all entries are zero except the diagonal ones which are 1). It satisfies  $AI = A$  and  $IA = A$ .
- The associative law  $A(BC) = (AB)C$  holds. Hence, we can write  $ABC$  without ambiguity.
- The distributive laws including  $A(B + C) = AB + AC$  hold.  $\diamond$

**Example 97.**  $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ , so we have no commutative law.  $\diamond$

The **inverse**  $A^{-1}$  of a matrix  $A$  is characterized by  $A^{-1}A = I$  and  $AA^{-1} = I$ .

**Example 98.** You can check that  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . Useful to remember!  $\diamond$

**Example 99.** Equations like  $7x_1 - 2x_2 = 3$ ,  $2x_1 + x_2 = 4$  can be equivalently expressed as  $\begin{pmatrix} 7 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ . Multiplying (from the left!) by  $\begin{pmatrix} 7 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -2 & 7 \end{pmatrix}$  produces  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 1 & 2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  which gives the solution of the original equations.  $\diamond$

The **determinant** of  $A$ , written as  $\det(A)$  or  $|A|$ , is a number with the property that:

$$\begin{aligned} \det(A) \neq 0 &\iff A \text{ is invertible} \\ &\iff A\mathbf{x} = \mathbf{b} \text{ has a (unique) solution } \mathbf{x} \text{ (for all } \mathbf{b}) \\ &\iff A\mathbf{x} = \mathbf{0} \text{ is only solved by } \mathbf{x} = \mathbf{0} \end{aligned}$$

**Example 100.**  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$  which appeared in the formula for the inverse.  $\diamond$

We will compute determinants of larger matrices next time.



**Review.** properties of determinants

◇

The **determinant** of any matrix can be computed by picking a row  $i$  and calculating  $\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{[i,j]})$ , where  $A_{[i,j]}$  is obtained from  $A$  by deleting the  $i$ th row and  $j$ th column.

The determinant satisfies  $\det(A^T) = \det(A)$  (as a consequence, we can adjust the above formula to expand along columns instead of along a row) and  $\det(AB) = \det(A)\det(B)$ .

**Example 101.**  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

◇

Vectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$  are (linearly) **independent** if  $c_1\mathbf{x}_1 + \dots + c_n\mathbf{x}_n = 0$  only for  $c_1 = c_2 = \dots = c_n = 0$ .

When checking independence of  $n$  many  $n \times 1$  column vectors, we can use determinants! They are independent if and only if  $\det(\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n) \neq 0$ . [Do you see why?!]

**Example 102.** Are the vectors  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  independent?

**Solution.**  $\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -3 \neq 0$ . Hence the vectors are independent.

◇

**Example 103.** Are the vectors  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$  independent?

**Solution.**  $\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 0 \end{pmatrix} = 2\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - (-1)\det \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = -2 + 2 = 0$ . Hence the vectors are dependent.

**Solution.**  $4\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$ . So, by definition, they are dependent.

◇

**Review.** We took another look at Theorem 59 from Lecture 13 to convince ourselves that it follows from what we just learned about determinants.

◇

**Example 104.**  $x'' - x' - 2x = 0$  has solutions  $x_1 = e^{2t}, x_2 = e^{-t}$ .

Since  $W(t) = \det \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}$  and so  $W(0) = \det \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = -3 \neq 0$ , our two solutions are indeed independent. As a consequence, the general solution is  $c_1x_1 + c_2x_2$ .

◇

**Example 105.** Introducing  $y = x'$ , the previous DE is equivalent to the first-order system  $x' = y, y' = 2x + y$ . Our known solutions translate into  $x_1 = e^{2t}, y_1 = 2e^{2t}$  and  $x_2 = e^{-t}, y_2 = -e^{-t}$ .

Writing  $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ , this system is  $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{x}$  with solutions  $\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ .

For fixed  $t$ , these two vectors are independent if and only if  $\det(\mathbf{x}_1 \ \mathbf{x}_2) = \det \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix} \neq 0$ . Note that this is precisely the Wronskian of the previous example.

◇

As in this last example, if  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  are solutions to  $\mathbf{x}' = A(t)\mathbf{x}$  (a homogeneous linear first-order system of DEs), then their **Wronskian** is the determinant  $W(t) = \det(\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n)$ . Next time, we will see the expected properties it again has.

**Theorem 106.** Let  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  be solutions of  $\mathbf{x}' = A(t)\mathbf{x}$ .  $A(t)$  is  $n \times n$ , entries continuous on  $I$ .

$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$  is the general solution

$\iff \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  are independent

$\iff$  the **Wronskian**  $W(t) = \det(\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n) \neq 0$  for  $t \in I$  of our choice

Moreover, such solutions always exist (on all of  $I$ ).

**Example 107.**  $x''' - 6x'' + 11x' - 6x = 0$  has solutions  $x_1 = e^t$ ,  $x_2 = e^{2t}$ ,  $x_3 = e^{3t}$ .

$$W(t) = \det \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}, W(0) = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} = \det \begin{pmatrix} 2 & 3 \\ 4 & 9 \end{pmatrix} - \det \begin{pmatrix} 1 & 3 \\ 1 & 9 \end{pmatrix} + \det \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = 2 \neq 0.$$

This certifies<sup>15</sup> that our three solutions are indeed independent. As a consequence, the general solution is  $c_1x_1 + c_2x_2 + c_3x_3$ .  $\diamond$

**Example 108.** Introducing  $y = x'$  and  $z = x''$ , the previous DE is equivalent to the first-order system  $x' = y$ ,  $y' = z$ ,  $z' = 6x - 11y + 6z$ . Writing  $\mathbf{x} = (x, y, z)^T$ , this system can be expressed as

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \mathbf{x}.$$

The solutions from the previous example translate into  $\mathbf{x}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$ ,  $\mathbf{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$ ,  $\mathbf{x}_3 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}$ .

The Wronskian is  $W(t) = \det(\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3) = \det \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}$ . Exactly as before!

We again conclude that  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  are independent. The general solution is  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$ .  $\diamond$

**Example 109.** Solve the initial value problem  $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix} \mathbf{x}$ ,  $\mathbf{x}(0) = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ .

**Solution.** From above, we know that the general solution is  $\mathbf{x} = c_1 \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + c_3 \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ . The matrix  $(\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3)$  is called a **fundamental matrix**.

In order to solve the IVP, we have to solve  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ .

$$\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & -1 \\ 1 & 4 & 9 & 3 \end{array} \implies \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 5 \end{array} \implies \begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{array} \implies c_3 = 1, c_2 = -1, c_1 = -2$$

Hence, the solution to the IVP is  $\mathbf{x}(t) = -2 \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} - \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = \begin{pmatrix} -2e^t - e^{2t} + e^{3t} \\ -2e^t - 2e^{2t} + 3e^{3t} \\ -2e^t - 4e^{2t} + 9e^{3t} \end{pmatrix}$ .  $\diamond$

We now turn to actually solving systems  $\mathbf{x}' = A\mathbf{x}$  where  $A$  is a  $n \times n$  matrix with constant entries.

Looking back at our examples so far, it makes sense to look for solutions of the form  $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$  with  $\mathbf{v}$  a vector which does not depend on  $t$ . Plugging into the DE, we get  $\mathbf{x}' = \mathbf{v}\lambda e^{\lambda t} \stackrel{!}{=} A\mathbf{v}e^{\lambda t}$ . Cancelling the exponentials, we see that we have a solution if and only if  $A\mathbf{v} = \lambda\mathbf{v}$ .

For those familiar with the language of Linear Algebra this means that  $\mathbf{v}$  is an **eigenvector** of  $A$  with **eigenvalue**  $\lambda$ .

<sup>15</sup> Though we do know *a priori* that our method of solving HLDEs with constant coefficients will always produce independent solutions. It is good to see that the Wronskian agrees; it has to.

**Review.** solutions to the system discussed last time

◇

In order to solve  $\mathbf{x}' = A\mathbf{x}$ , we look for solutions  $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ .

Plugging into the DE, we get  $\mathbf{x}' = \mathbf{v}\lambda e^{\lambda t} \stackrel{!}{=} A\mathbf{v}e^{\lambda t}$ . Cancelling the exponentials, we see that we have a solution if and only if  $A\mathbf{v} = \lambda\mathbf{v}$ .

**Definition 110.** If  $A\mathbf{v} = \lambda\mathbf{v}$ , for  $\mathbf{v} \neq 0$ , then  $\mathbf{v}$  is an **eigenvector** of  $A$  with **eigenvalue**  $\lambda$ .

In order to find these, note that  $A\mathbf{v} = \lambda\mathbf{v} = \lambda I\mathbf{v}$  is equivalent to  $(A - \lambda I)\mathbf{v} = 0$ . By the properties of determinants, this is only possible if  $\det(A - \lambda I) = 0$ . This determinant is a polynomial in  $\lambda$ , the **characteristic polynomial** of  $A$ . Its roots are the eigenvalues  $\lambda$ .

For a specific eigenvalue  $\lambda$ , we then solve  $(A - \lambda I)\mathbf{v} = 0$  to find the eigenvector(s)  $\mathbf{v}$ .

**Example 111.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial is

$$\det \left[ \begin{pmatrix} 4 & 2 \\ -3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = \det \begin{pmatrix} 4-\lambda & 2 \\ -3 & -1-\lambda \end{pmatrix} = (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2).$$

This means that the eigenvalues are  $\lambda = 1$  and  $\lambda = 2$ .

**$\lambda = 1$ .** To find  $\mathbf{v}$ , we have to solve  $\begin{pmatrix} 3 & 2 \\ -3 & -2 \end{pmatrix} \mathbf{v} = 0$ . We find  $\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ , or any multiple thereof.

If that was too fast for you, note that we need to solve  $3v_1 + 2v_2 = 0$ ,  $-3v_1 - 2v_2 = 0$ . The second equation is worthless; it is just the first one times  $-1$ . Hence, we are free to set, for instance,  $v_1 = c$ . The equations then give  $v_2 = -\frac{3}{2}c$ . The most general solution therefore is  $\mathbf{v} = (c, -\frac{3}{2}c)^T$ . Our eigenvector above is the choice  $c = 2$ . [Why do we not care about which multiple of the eigenvector to pick?]

**$\lambda = 2$ .** Now, we have to solve  $\begin{pmatrix} 2 & 2 \\ -3 & -3 \end{pmatrix} \mathbf{v} = 0$ . We find  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Consequently, we have solutions  $\mathbf{x}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^t$  and  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ .

Let us check that these are independent using the Wronskian.

$$W(t) = \det \begin{pmatrix} 2e^t & e^{2t} \\ -3e^t & -e^{2t} \end{pmatrix}, \quad W(0) = \det \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} = 1 \neq 0.$$

This certifies that  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are independent.

Therefore, the general solution is  $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} = \begin{pmatrix} 2c_1e^t + c_2e^{2t} \\ -3c_1e^t - c_2e^{2t} \end{pmatrix}$ .

◇

**Remark 112. (JustForFun)**

Recall that, abstractly, vectors are anything that can be added and scaled. In that abstract sense, matrices are functions (better, operators), which take vectors as input and return vectors as output (a matrix  $A$  takes the vector  $\mathbf{x}$  and returns the vector  $A\mathbf{x}$ ). These operators are linear, meaning that, for instance,  $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ . Now, think of (differentiable) functions on the real line. They can be added and scaled, and so form a vector space. There is a very interesting and basic linear operator: the derivative  $D$ , which takes a function  $f$  and returns  $Df = f'$ .

What are the eigenfunctions<sup>16</sup>  $f$  and eigenvalues  $\lambda$  of  $D$ ? That is, what are the solutions to  $Df = \lambda f$ ? For any  $\lambda$ , there is a solution: the exponential  $f = e^{\lambda x}$  (or multiples thereof).

In other words, any  $\lambda$  is an eigenvalue of  $D$  and  $e^{\lambda x}$  is a corresponding eigenfunction. In short, *the exponentials are important because they are the eigenfunctions of the derivative!*

◇

16. That is just a more politically correct name for eigenvector in this context.

**Review.** eigenvectors, eigenvalues and corresponding solutions to systems ◇

**Example 113.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} -5 & 3 & 3 \\ 0 & -1 & 2 \\ -6 & 5 & 2 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial is

$$\det \begin{pmatrix} -5-\lambda & 3 & 3 \\ 0 & -1-\lambda & 2 \\ -6 & 5 & 2-\lambda \end{pmatrix} = (-1-\lambda) \det \begin{pmatrix} -5-\lambda & 3 \\ -6 & 2-\lambda \end{pmatrix} - 2 \det \begin{pmatrix} -5-\lambda & 3 \\ -6 & 5 \end{pmatrix} = \dots = -\lambda^3 - 4\lambda^2 - \lambda + 6,$$

which has roots  $\lambda = 1, -2, -3$ . These are the eigenvalues.

$$\lambda = 1. \quad \begin{pmatrix} -6 & 3 & 3 \\ 0 & -2 & 2 \\ -6 & 5 & 1 \end{pmatrix} \mathbf{v} = 0. \text{ We eliminate: } \begin{array}{ccc|c} -6 & 3 & 3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \end{array} \implies \begin{array}{ccc|c} -6 & 3 & 3 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

The zero row signifies that the original three rows were dependent (they have to be, because the determinant is zero!). We really only have two equations for our three unknowns  $v_1, v_2, v_3$ . We are free to set, for instance,  $v_3 = c$ . Then the second equation ( $-2v_2 + 2v_3 = 0$ ) implies  $v_2 = c$ . Finally, the first equation ( $-6v_1 + 3v_2 + 3v_3 = 0$ ) implies  $v_1 = c$ . The most general solution to the eigenvector equation therefore is  $\mathbf{v} = (c, c, c)^T$ . Since we don't care about multiples, we choose  $\mathbf{v} = (1, 1, 1)^T$ .

$$\lambda = -2. \quad \begin{pmatrix} -3 & 3 & 3 \\ 0 & 1 & 2 \\ -6 & 5 & 4 \end{pmatrix} \mathbf{v} = 0. \text{ We eliminate: } \begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array} \text{ and see that the third equation is redundant.}$$

Let  $v_3 = c$ . Then,  $v_2 = -2c$  (from the second equation). Finally,  $-3v_1 + 3v_2 + 3v_3 = 0$  implies  $v_1 = -c$ . Thus,  $\mathbf{v} = (-c, -2c, c)^T$  and we choose, for instance,  $\mathbf{v} = (1, 2, -1)^T$ .

$$\lambda = -3. \quad \begin{pmatrix} -2 & 3 & 3 \\ 0 & 2 & 2 \\ -6 & 5 & 5 \end{pmatrix} \mathbf{v} = 0. \text{ We eliminate: } \begin{array}{ccc|c} -2 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -4 & -4 & 0 \end{array} \text{ and see that the third equation is redundant}^{17}.$$

Let  $v_3 = c$ . Then,  $v_2 = -c$ . Finally,  $-2v_1 + 3v_2 + 3v_3 = -2v_1 = 0$  implies  $v_1 = 0$ . Thus,  $\mathbf{v} = (0, -c, c)^T$  and we choose, for instance,  $\mathbf{v} = (0, 1, -1)^T$ .

Consequently, we have found solutions  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$ ,  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} e^{-2t}$ ,  $\mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-3t}$ .

They are independent (you could use the Wronskian to check!) as will always be the case if we use our approach appropriately to solve systems with constant coefficients. ◇

As in the case of one-dimensional equations, there are two issues we will have to think and worry about. Firstly, an eigenvalue may be a repeated root of the characteristic polynomial and, secondly, we may encounter complex roots. The case of complex roots is simple to address, the first issue, however, will cause some headaches.

<sup>17</sup>. With some practice, you can decide right away that the third (or, in fact, any one of the three) is redundant, that is, a linear combination of the other two. You are ready for that shortcut when you know how to explain it!

**Review.** complex numbers  $z = x + iy = re^{i\theta}$ , **conjugate**  $\bar{z} = x - iy = re^{-i\theta}$  ◇

Note that  $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$ . In particular,  $z\bar{z}$  is always real.

**Example 114.**  $\frac{1}{3-4i} = \frac{3+4i}{(3-4i)(3+4i)} = \frac{3+4i}{3^2+4^2} = \frac{3}{25} + \frac{4}{25}i$  ◇

In general,  $\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$ . Hence, we can do algebra using complex numbers. For instance, we can do elimination to solve linear equations involving complex numbers.

Note that the **real part**  $\operatorname{Re}(z) = x = \frac{1}{2}(z + \bar{z})$  and the **imaginary part**  $\operatorname{Im}(z) = y = \frac{1}{2i}(z - \bar{z})$  can each be written as a linear combination of  $z$  and its conjugate. Since complex solutions to homogeneous linear DEs come in conjugate pairs, we have the following principle.

**Theorem 115.** If  $\mathbf{x}(t)$  solves  $\mathbf{x}' = A(t)\mathbf{x}$ , where  $A$  is  $n \times n$  with real entries, then  $\operatorname{Re}(\mathbf{x}(t))$  and  $\operatorname{Im}(\mathbf{x}(t))$  are solutions as well.

**Example 116.** If  $\mathbf{x}(t) = \begin{pmatrix} 2-3i \\ i \\ 1+i \end{pmatrix} e^{(2+5i)t}$ , then we use  $e^{(2+5i)t} = e^{2t}(\cos(5t) + i\sin(5t))$  to find  $\operatorname{Re}(\mathbf{x}(t)) = e^{2t} \begin{pmatrix} 2\cos(5t) + 3\sin(5t) \\ -\sin(5t) \\ \cos(5t) - \sin(5t) \end{pmatrix}$  and  $\operatorname{Im}(\mathbf{x}(t)) = e^{2t} \begin{pmatrix} 2\sin(5t) - 3\cos(5t) \\ \cos(5t) \\ \sin(5t) + \cos(5t) \end{pmatrix}$ . ◇

**Example 117.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial  $\det \begin{pmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) + 5 = \lambda^2 + 4$  has roots  $\pm 2i$ .

To find the eigenvector for  $\lambda = 2i$ , we solve  $\begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \mathbf{v} = 0$ .

Depending on which row we look at, we can “see” either  $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$  or  $\mathbf{v}_2 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$ . At least, we can check that they are both eigenvectors. We can also check that they are just multiples of each other:  $\mathbf{v}_1 = (1-2i)\mathbf{v}_2$ .

Alternatively, we could just do elimination: by subtracting  $\frac{1}{1-2i} = \frac{1}{5} + \frac{2}{5}i$  times the first row from the second row,  $\begin{pmatrix} 1-2i & -5 \\ 0 & 0 \end{pmatrix} \mathbf{v} = 0$ . Note that we had to get the zero row. Why!? Thus we are free to set  $v_2 = c$ , and the first equation  $((1-2i)v_1 - 5v_2 = 0)$  gives us  $v_1 = \frac{5}{1-2i}c = (1+2i)c$ . Hence, the most general solution to the eigenvector equation is

$$\mathbf{v} = \begin{pmatrix} (1+2i)c \\ c \end{pmatrix},$$

and we observe that  $\mathbf{v}_2$  is the choice  $c = 1$  while  $\mathbf{v}_1$  is the choice  $c = 1 - 2i$ .

We do not need to find the eigenvector for  $\lambda = -2i$  (it will just be the conjugate of the eigenvector we just found).

Finally, we split the complex solution  $\mathbf{x} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} e^{2it}$  into real and imaginary part:

$$\operatorname{Re}(\mathbf{x}) = \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix}, \operatorname{Im}(\mathbf{x}) = \begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix}$$

The general solution is  $c_1 \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix} = \begin{pmatrix} (c_1 + 2c_2)\cos(2t) + (c_2 - 2c_1)\sin(2t) \\ c_1\cos(2t) + c_2\sin(2t) \end{pmatrix}$ . ◇

**Remark 118.** Note that solutions can look different, while being equivalent. For instance, in the previous example, we could have instead chosen the complex solution  $\tilde{\mathbf{x}} = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix} e^{2it}$ . Then,  $\operatorname{Re}(\tilde{\mathbf{x}}) = \begin{pmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{pmatrix}$ ,  $\operatorname{Im}(\tilde{\mathbf{x}}) = \begin{pmatrix} 5\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}$ . That these solutions generate the same general solution follows because  $\operatorname{Re}(\tilde{\mathbf{x}}) = \operatorname{Re}(\mathbf{x}) + 2\operatorname{Im}(\mathbf{x})$  and  $\operatorname{Im}(\tilde{\mathbf{x}}) = \operatorname{Im}(\mathbf{x}) - 2\operatorname{Re}(\mathbf{x})$ . Both of these are consequences of  $\tilde{\mathbf{x}} = (1-2i)\mathbf{x}$ ; can you see that?! ◇

**Example 119.** Solve  $\mathbf{x}' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial  $(4 - \lambda)^2 + 9$  has roots  $4 \pm 3i$ .

To find the eigenvector for  $\lambda = 4 + 3i$ , we solve  $\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix} \mathbf{v} = 0$ . We find  $\mathbf{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ .

The complex solution  $\mathbf{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(4+3i)t}$  has real part  $\operatorname{Re}(\mathbf{x}) = e^{4t} \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$  and imaginary part  $\operatorname{Im}(\mathbf{x}) = e^{4t} \begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}$ . The corresponding fundamental matrix is  $X(t) = e^{4t} \begin{pmatrix} \cos(3t) & \sin(3t) \\ \sin(3t) & -\cos(3t) \end{pmatrix}$ .

[The fundamental matrix satisfies  $X' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} X$ . Why?!]

◇

**Remark 120. (JustForFun)** Let  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Then we easily check that  $J^2 = -I$ , where  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  is the identity matrix. If we think of  $I$  as a matrix version of the number 1, then  $J$  is a matrix version of the imaginary unit  $i$ . More generally, we can then associate a complex number  $x + iy$  with the matrix  $xI + yJ = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ . Addition and multiplication work equally on both sides of this identification; why?! Division works equally as well:  $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{x^2 + y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$  corresponds to  $(x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}$ .

The reason that complex numbers can be associated with  $2 \times 2$  matrices is that multiplying a point in the plane (represented by a complex number  $se^{i\varphi}$ ) with a complex number  $re^{i\theta}$  has a geometric meaning (which identifies it as a linear transformation): the point gets scaled by the factor  $r$  and rotated by  $\theta$  (to get  $rse^{i(\theta+\varphi)}$ ).

Since  $e^{i\theta}$  is just rotation by  $\theta$ , the corresponding matrix  $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$  is the rotation matrix that you have probably seen before. Now, you understand why it takes this form! Also, you know that things look simpler if we let complex numbers represent these matrices. That's something people working in computer graphics do!

[This works for 2 dimensions, so is there something<sup>18</sup> like the complex numbers for 3 dimensions? It turns out that the answer is no. However, as discovered by Hamilton, in four dimensions one finds the quaternions  $x + iy + jz + kw$  with the rules  $i^2 = -1$ ,  $j^2 = -1$ ,  $k^2 = -1$ ,  $ijk = -1$ . They satisfy all the usual rules, except that commutativity is lost (and the quaternions are the only such construction). Quaternions are indeed used in computer graphics!]

By the way, with this understanding, we can give a quick (but somewhat unjustified) solution to the previous example (note the special form of the involved matrix). To solve  $X' = (4I + 3J)X$ , where all quantities are  $2 \times 2$  matrices, we interpret them as complex numbers to get  $X(t) = e^{(4+3i)t} = e^{4t}(\cos(3t) + i\sin(3t))$  which we then reinterpret as the matrix  $X(t) = e^{4t} \begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}$ . This agrees perfectly with our previous solution (which only differs in the sign for the second column).

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**Remark 121. (JustForFun)** The sequence  $(x_0, x_1, x_2, \dots) = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots)$  is world-famous and known as the [Fibonacci sequence](#). It is defined by  $x_{n+1} = x_n + x_{n-1}$  (the recurrence; a discrete analog of a differential equation) and  $x_0 = 0$ ,  $x_1 = 1$  (the initial conditions).

Just as we did for differential equations, we can convert the second-order recurrence into a system of first-order equations:  $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_n + x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$

Iterating, we get  $\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = \dots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .

Note that if  $\mathbf{v}$  is an eigenvector of  $A$ , then  $A^n \mathbf{v}$  is easy to compute; it is just  $A^n \mathbf{v} = \lambda^n \mathbf{v}$  where  $\lambda$  is the eigenvalue.

The characteristic polynomial of  $A$  is  $\lambda^2 - \lambda - 1$ , and the eigenvalues are  $\lambda_1 = \frac{1 + \sqrt{5}}{2} \approx 1.618$  (the golden mean!) and  $\lambda_2 = \frac{1 - \sqrt{5}}{2} \approx -0.618$ . Let  $\mathbf{v}_1, \mathbf{v}_2$  be the corresponding eigenvectors<sup>19</sup> (which we could compute; but which we will not need for the conclusion we want to make).

Write<sup>20</sup>  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$ . Then  $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1^n c_1 \mathbf{v}_1 + \lambda_2^n c_2 \mathbf{v}_2 \approx \lambda_1^n c_1 \mathbf{v}_1$  (because  $|\lambda_2| < 1$ ).

Thus, approximately, increasing  $n$  by 1 results in multiplication by  $\lambda_1 \approx 1.618$ .

Indeed,  $\frac{x_{n+1}}{x_n} \approx \lambda_1$  is witnessed even by the early examples  $\frac{34}{21} \approx 1.61905$ ,  $\frac{55}{34} \approx 1.61765$ ,  $\frac{89}{55} \approx 1.61818$ .

◇

18. We are looking for  $i$  and  $j$ , together with some rules like  $i^2 = -1$ , such that the “numbers”  $x + iy + jz$  can be added and multiplied as usual with the important requirement that every element except 0 should be invertible.

19.  $\mathbf{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ ,  $\mathbf{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}$  would do.

20. If we did the calculations with the above choice of  $\mathbf{v}_1, \mathbf{v}_2$ , we would find  $c_1 = \frac{5 + \sqrt{5}}{10}$ ,  $c_2 = \frac{5 - \sqrt{5}}{10}$ .

## Review — currently, all DEs considered are linear

- Homogeneous linear DEs of order  $n$ :  $Ly = 0$ 
  - constant coefficients
 

using the roots of the characteristic polynomial of  $L$ , we have a complete recipe for finding  $n$  independent solutions (and we know how to deal with complex and repeated roots)
  - non-constant coefficients
 

we have no method for solving such equations (except first-order DEs by integrating factor); however, we know existence of  $n$  independent solutions; moreover, if we are handed  $n$  prospective solutions, then we can determine whether these are independent solutions (plugging into the DE to check that they actually solve, and then using the Wronskian to check independence)
- Inhomogeneous linear DEs:  $Ly = f$ 

first of all, we know that if we find a single solution  $y_p$  then we get the general solution by adding the solutions of the homogeneous equation

  - constant coefficients plus suitable  $f$  (namely,  $f$  solves a const coeff eq  $\tilde{L}f = 0$ )
 

by combining the roots of  $L$  (the “old” ones) with the roots of  $\tilde{L}$  (the “new” ones), we have a recipe to find  $y_p$ ; namely, since  $\tilde{L}Ly_p = 0$ , there has to be a  $y_p$  that is a combination of the “new” solutions; once we have the shape of  $y_p$  with undetermined coefficients, we need to plug into the DE to find these coefficients
  - non-constant coefficients
 

once we know the general solution of the homogeneous equation, we can use variation of constants to find  $y_p$ ; we have only discussed the second-order case, for which we have derived a formula in terms of integrals involving two independent solutions  $y_1, y_2$  of  $Ly = 0$ ; this is one the few (the only?) formulas that you should memorize for the test (deriving takes too long)
- Homogeneous systems of linear DEs:  $\mathbf{x}' = A(t)\mathbf{x}$ , where  $A(t)$  is an  $n \times n$  matrix
 

we know that any (linear) DE of order  $n$  can be written as a  $n \times n$  (linear) first-order system

  - constant coefficients (that is,  $A$  does not depend on  $t$ )
 

we again have a recipe for finding  $n$  independent solutions; namely, we find the eigenvalues  $\lambda$  as the roots of the characteristic polynomial  $\det(A - \lambda I)$  and then find corresponding eigenvectors  $\mathbf{v}$ ; each pair gives us a solution  $\mathbf{v}e^{\lambda t}$ ; we do not yet know how to deal with complex and repeated eigenvalues
  - non-constant coefficients (note how our knowledge matches the case of HLDEs of order  $n$ )
 

we have no method for solving such equations; however, we know existence of  $n$  independent solutions; moreover, if we are handed  $n$  prospective solutions, then we can determine whether these are independent solutions (plugging into the DE to check that they actually solve, and then using the Wronskian to check independence)
- Mechanical vibrations
  - $mx'' + kx = 0$  describes oscillations of a mass  $m$  on a spring with spring constant  $k$  that's the undamped case; for these, and other oscillations, we know how to determine amplitude and frequency (using that  $A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cos(\omega t - \alpha)$ )
  - $mx'' + cx' + kx = 0$  models damped motion ( $c > 0$  is the damping coefficient)
 

solutions can take three different forms:  $Ae^{-\rho t} \cos(\omega t - a)$  (underdamped),  $Ae^{-\rho t} + Be^{-\rho t}$  (overdamped), or  $(A + Bt)e^{-\rho t}$  (critically damped)
  - $mx'' + cx' + kx = f(x)$  models addition of an external force (usually periodic)
 

if  $c = 0$  then there is the possibility of resonance if natural and external frequency match; if  $c > 0$  then we might still have practical resonance; also, if  $c > 0$  (and  $f$  is periodic), then solutions  $x$  split into  $x = x_{\text{sp}} + x_{\text{tr}}$ , the steady periodic oscillations  $x_{\text{sp}}$  and the transient motion  $x_{\text{tr}}$



**Remark 122.** One of the problems on the midterm asked for solving  $y''' - y = e^x + 7$ .

The characteristic polynomial  $r^3 - 1$  has root  $r = 1$ . To find the other roots, we can do polynomial division to get  $r^3 - 1 = (r - 1)(r^2 + r + 1)$ .

More generally, the roots of  $z^n - 1$  are called  $n$ -th **roots of unity**.  $z^n = 1$  implies that  $|z| = 1$ , which means that these numbers lie on the unit circle. In particular, they are of the form  $e^{i\theta}$ . Remembering that  $e^{2\pi i} = 1$ , we find that  $\zeta = e^{2\pi i/n}$  is a  $n$ -th root of unity and so are  $\zeta^2 = e^{4\pi i/n}$ ,  $\zeta^3 = e^{6\pi i/n}$ , ...

Geometrically, the  $n$ -th roots of unity form the vertices of a regular  $n$ -gon. Now, go back to the equation  $z^3 - 1$  and mark the solutions on the unit circle.  $\diamond$

**Remark 123.** Another problem on the midterm asked to find a homogeneous linear DE solved by solutions of the inhomogeneous linear DE  $y'' + xy = e^x$ .

Note that this DE does not have constant coefficients. Yet, we can proceed as we did in the case of constant coefficients:  $e^x$  solves a HLDE with constant coefficients and root 1 (the “new” root); this is another way of saying that  $(D - 1)e^x = 0$ . Applying  $D - 1$  to both sides of the DE, we get  $(D - 1)(y'' + xy) = y''' - y'' + xy' + (1 - x)y = 0$ , which is a homogeneous linear DE.

Just one word of caution: we can write the initial DE as  $(D^2 + x)y = e^x$ ; however, we need to be careful when working with differential operators which involve both  $D$  and  $x$ . That’s because  $Dx \neq xD$ , which we can see from  $Dxy = xy' + y$  versus  $x Dy = xy'$ . In other words,  $x$  and  $D$  don’t commute (just like generic matrices).  $\diamond$

**Example 124.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1-\lambda \\ -1 & 1 \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) - 2.$$

Since  $x^3 - 3x - 2 = (x + 1)^2(x - 2)$ , the eigenvalues are  $\lambda = 1 - x = 2, 2, -1$ . Note that  $\lambda = 2$  is repeated! We say that the eigenvalue  $\lambda = 2$  has **multiplicity 2**.

$$\lambda = -1. \quad \begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array} \xRightarrow[2r_3+r_1]{2r_2-r_1} \begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array} \xRightarrow{} \begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

Setting,  $v_3 = c$ , we find  $v_2 = -c$ . Then,  $2v_1 + v_2 - v_3$  implies  $v_1 = c$ . Setting  $c = 1$ , we find  $\mathbf{v} = (1, -1, 1)^T$ .

$$\lambda = 2. \quad \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array} \xRightarrow[r_3-r_1]{r_2+r_1} \begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

The two zero rows are good news! It means that we will find two independent eigenvectors.

Indeed, we are free to set  $v_3 = c$  and  $v_2 = d$ . Since  $-v_1 + v_2 - v_3 = 0$ , it follows that  $v_1 = d - c$ . Hence, the most general solution to the eigenvector equation is

$$\mathbf{v} = \begin{pmatrix} d - c \\ d \\ c \end{pmatrix} = c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + d \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have found solutions  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}$ ,  $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$ ,  $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$ .

The Wronskian at 0 is  $W(0) = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -3 \neq 0$ , which certifies that our three solutions are independent.

Hence, the general solution is  $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$ .  $\diamond$



**Example 125.** Consider  $\mathbf{x}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \mathbf{x}$ .

The characteristic polynomial  $(1 - \lambda)(7 - \lambda) + 9 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$  has the double root  $\lambda = 4$ .

However,  $\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \mathbf{v} = 0$  has solution only  $\mathbf{v} = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

We say that the eigenvalue 4 is **defective** with **defect** 1 (number of missing eigenvectors).

So far, we have found the solution  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$  but we are missing a second independent solution.  $\diamond$

We want to solve  $\mathbf{x}' = A\mathbf{x}$ . Suppose that  $\lambda$  is a repeated and defective eigenvalue.

- As a first attempt, we might try to look for a solution of the form  $\mathbf{x} = \mathbf{w}te^{\lambda t}$ . Plugging into the DE, we get  $\mathbf{x}' = \mathbf{w}e^{\lambda t} + \mathbf{w}\lambda te^{\lambda t} \stackrel{!}{=} A\mathbf{x} = A\mathbf{w}te^{\lambda t}$ . Setting  $t = 0$ , this implies  $\mathbf{w} = 0$  which means our first attempt failed.
- Not giving up, we next look for a solution of the form  $\mathbf{x} = \mathbf{u}e^{\lambda t} + \mathbf{w}te^{\lambda t}$ . Plugging into the DE, we now get  $\mathbf{x}' = \mathbf{u}\lambda e^{\lambda t} + \mathbf{w}e^{\lambda t} + \mathbf{w}\lambda te^{\lambda t} \stackrel{!}{=} A\mathbf{x} = A\mathbf{u}e^{\lambda t} + A\mathbf{w}te^{\lambda t}$ . Equating coefficients, we find  $A\mathbf{w} = \lambda\mathbf{w}$  and  $A\mathbf{u} = \lambda\mathbf{u} + \mathbf{w}$ . Equivalently,  $(A - \lambda I)\mathbf{w} = 0$  and  $(A - \lambda I)\mathbf{u} = \mathbf{w}$ . [ $\mathbf{u}$  will be called a generalized eigenvector of rank 2.]

**Example. (cont'd)** Let us find  $\mathbf{u}$  such that  $\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Note that the second equation is  $-1$  times the first. Hence, setting  $w_2 = c$ , we get  $w_1 = -c - \frac{1}{3}$ . Any choice of  $c$  will give us a vector  $\mathbf{w}$  that we need to construct a second solution. For instance, choosing  $c = 0$ , we get  $\mathbf{w} = (-1/3, 0)^T$ . This means that  $\mathbf{x}_2 = \left[ \begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} \right] e^{4t}$  is a second independent solution of the DE.  $\diamond$

This approach works whenever we have a defective eigenvalue of multiplicity 2.

The same idea leads to the concept of generalized eigenvectors.

**Definition 126.**  $\mathbf{v}_1, \dots, \mathbf{v}_k$  form a **chain of generalized eigenvectors** for the eigenvalue  $\lambda$  if

$$\begin{aligned} (A - \lambda I)\mathbf{v}_1 &= 0 && [\text{solution of } \mathbf{x}' = A\mathbf{x}: \mathbf{v}_1 e^{\lambda t}] \\ (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1 && [\text{solution: } (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t}] \\ (A - \lambda I)\mathbf{v}_3 &= \mathbf{v}_2 && \left[ \text{solution: } \left( \mathbf{v}_1 \frac{t^2}{2} + \mathbf{v}_2 t + \mathbf{v}_3 \right) e^{\lambda t} \right] \\ &\vdots && \\ (A - \lambda I)\mathbf{v}_k &= \mathbf{v}_{k-1} && \left[ \text{solution: } \left( \mathbf{v}_1 \frac{t^{k-1}}{(k-1)!} + \mathbf{v}_2 \frac{t^{k-2}}{(k-2)!} + \dots \mathbf{v}_{k-1} t + \mathbf{v}_k \right) e^{\lambda t} \right] \end{aligned}$$

Some comments on generalized eigenvectors:

- Note that  $\mathbf{v}_k$  satisfies  $(A - \lambda I)^k \mathbf{v}_k = 0$  but  $(A - \lambda I)^{k-1} \mathbf{v}_k = \mathbf{v}_1 \neq 0$ . We say  $\mathbf{v}_k$  is a generalized eigenvector of **rank**  $k$ .
- The vectors in several chains are independent if and only if the chains are based on independent eigenvectors (the  $\mathbf{v}_1$ 's).
- For every  $n \times n$  matrix  $A$ , we can find  $n$  independent generalized eigenvectors. In particular, we can then find the general solution of  $\mathbf{x}' = A\mathbf{x}$  by constructing the corresponding solutions as indicated above.

**Review.** generalized eigenvectors and corresponding solutions to  $\mathbf{x}' = A\mathbf{x}$

Recipe for solving  $\mathbf{x}' = A\mathbf{x}$ :

- find eigenvalues  $\lambda$
- for each  $\lambda$ , find eigenvectors
- if  $\lambda$  is defective, find enough chains<sup>21</sup>
- if  $\lambda = a \pm bi$  is complex, take real and imaginary part of the solutions found

◇

**Example 127.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial is  $\dots = -(\lambda + 1)^3$ .

Hence,  $\lambda = -1$  is an eigenvalue of multiplicity 3.

We first solve for eigenvectors:  $\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -5 & -2 & -7 & 0 \\ 1 & 0 & 1 & 0 \end{array} \xrightarrow[r_3 - r_1]{r_2 + 5r_1} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{array} \xrightarrow[3r_3 + r_2]{r_2/3} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$

Setting  $v_3 = c$ , we get  $v_2 = -c$  and then  $v_1 = -c$ . The choice  $c = -1$  gives  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ . The corresponding solution is  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}$ .

Since there was only one degree of freedom, there is no other independent eigenvector.  $\lambda = -1$  has **defect 2**.

Because there is only one eigenvector to build a chain upon, we now know that there has to be a chain  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  of three generalized eigenvectors.

To find  $\mathbf{v}_2$ , we need to solve  $\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ . We can save time and effort by reusing the elimination

we have already done:  $\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -5 & -2 & -7 & 1 \\ 1 & 0 & 1 & 0 \end{array} \xrightarrow[r_3 - r_1]{r_2 + 5r_1} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 6 \\ 0 & -1 & -1 & -2 \end{array} \xrightarrow[3r_3 + r_2]{r_2/3} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}$

This time,  $v_3 = c$  leads to  $v_2 = 2 - c$ .  $v_1 + v_2 + 2v_3 = 1$  then gives  $v_1 = -1 - c$ . Hence,  $\mathbf{v}_2 = \begin{pmatrix} -1 - c \\ 2 - c \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ . Note that the second summand is just an eigenvector! We can choose any  $c$ . For instance, choosing  $c = 0$  gives  $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$  with corresponding solution<sup>22</sup>  $\mathbf{x}_2 = \left[ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right] e^{-t}$ .

Finally, to find  $\mathbf{v}_3$  we have to solve  $(A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2$ . We can again reuse the elimination we have already done:

$\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -5 & -2 & -7 & 0 \\ 1 & 0 & 1 & 0 \end{array} \xrightarrow[r_3 - r_1]{r_2 + 5r_1} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{array} \xrightarrow[3r_3 + r_2]{r_2/3} \begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$

As before,  $v_3 = c$  leads to  $v_2 = -1 - c$ .  $v_1 + v_2 + 2v_3 = -1$  then gives  $v_1 = -c$ . Choosing  $c = 0$ , we get  $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

which gives us the solution<sup>23</sup>  $\mathbf{x}_3 = \left[ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] e^{-t}$ . ◇

**Example 128.** Suppose we have an eigenvalue  $\lambda$  of multiplicity 5.

Here are the 7 possibilities for chains, listed by the lengths of the chains that occur:

- (defect 0) 1, 1, 1, 1, 1 [i.e., 5 eigenvectors]
- (defect 1) 2, 1, 1, 1
- (defect 2) 2, 2, 1 or 3, 1, 1
- (defect 3) 3, 2 or 4, 1
- (defect 4) 5

Note that the defect is something we know (after computing the eigenvectors). We have seen how to do the defect 0 and defect 4 cases; the other ones are a little bit more intricate. ◇

21. These computations can become a bit intricate. For exams, we will content ourselves with the defective cases involving a single chain (per eigenvalue) only.

22. How does choosing a different  $c$  affect the solution  $\mathbf{x}_2$ . Why does it not make a difference?

23. Try and see what happens if you went looking for a fourth vector  $\mathbf{v}_4$  in the chain. Why does it fail?

**Review.** We now have a recipe to solve  $\mathbf{x}' = A\mathbf{x}$ . If  $A$  is  $n \times n$  (with constant entries), then we can find  $n$  independent solutions  $\mathbf{x}_1, \dots, \mathbf{x}_n$ .

In particular, the general solution is  $c_1\mathbf{x}_1(t) + \dots + c_n\mathbf{x}_n(t) = \Phi(t)\mathbf{c}$  where  $\Phi(t)$  is the **fundamental matrix**  $\Phi = (\mathbf{x}_1 \dots \mathbf{x}_n)$ .  $\diamond$

Let us solve (in a general abstract sense) the IVP  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \mathbf{x}_0$ .

If  $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ , then  $\mathbf{x}(0) = \Phi(0)\mathbf{c} \stackrel{!}{=} \mathbf{x}_0$ . We conclude that  $\mathbf{c} = \Phi(0)^{-1}\mathbf{x}_0$ . [Why is the matrix  $\Phi(0)$  invertible?!]

Hence,  $\mathbf{x}(t) = \Phi(t)\Phi(0)^{-1}\mathbf{x}_0$ . Note that  $\Phi(t)\Phi(0)^{-1}$  is another fundamental matrix. Why?!

**Theorem 129.** Let  $\Phi(t)$  be any fundamental matrix. Then  $\Phi(t)\Phi(0)^{-1} = e^{At}$ .

Here, the **matrix exponential**  $e^A$  is defined as  $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots$ . This is the same Taylor series which we know for the ordinary exponential.

Let us remind ourselves how the Taylor series  $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$  is connected with the simple differential equation satisfied by the exponential function.

Let us demonstrate, only using the Taylor series, that  $x(t) = e^{at}$  solves  $x' = ax$ . Indeed,

$$x'(t) = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{a^n t^n}{n!} = \sum_{n=1}^{\infty} \frac{a^n n t^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{a^n t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{a^{n+1} t^n}{n!} = a e^{at} = ax(t).$$

- Think about this: every step of this argument works equally well if the number  $a$  is replaced by a  $n \times n$  matrix  $A$ !
- In fact, this shows that  $e^{At}$  is a fundamental matrix of  $\mathbf{x}' = A\mathbf{x}$ .
- Clearly,  $e^{At}|_{t=0} = I$ . These two facts together actually prove our theorem above.

**Example 130.** Find a fundamental matrix for the (very easy) DE  $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}$ .

**Solution. (using  $e^{At}$ )** Let  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Note that  $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . A matrix such that  $A^n = 0$ , for some  $n$ , is called **nilpotent**. In a such a case, the infinite sum  $e^{At} = I + At + \frac{A^2 t^2}{2} + \dots$  is actually finite.

Here,  $e^{At} = I + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . This is a fundamental matrix.

**Solution. (using generalized eigenvectors)** The characteristic polynomial is  $\lambda^2$ . Hence,  $\lambda = 0$  is the only eigenvalue and has multiplicity 2. Solving the eigenvector equation  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , we (only) find  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (or any multiple). This means that  $\lambda = 0$  has defect 1. To find a generalized eigenvector  $\mathbf{v}_2$  of rank 2, we solve  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and obtain, for instance,  $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . The corresponding solutions are  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$ . This again reveals  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  as a fundamental matrix.

**Solution. (using nothing)** With  $\mathbf{x} = (x_1, x_2)^T$ , the system can also be written as  $x_1' = x_2$ ,  $x_2' = 0$ .

The second equation implies  $x_2 = c$  for a constant  $c$ . Then the first equation shows  $x_1 = ct + d$ .

Thus  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ct + d \\ c \end{pmatrix} = d \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} t \\ 1 \end{pmatrix}$ . Once more, we find  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  as a fundamental matrix.  $\diamond$

**Review.** If  $\mathbf{x}' = A\mathbf{x}$ , with  $A$  an  $n \times n$  matrix (with constant entries), then  $n$  independent solutions  $\mathbf{x}_1, \dots, \mathbf{x}_n$  can be combined into a **fundamental matrix**  $\Phi = (\mathbf{x}_1 \dots \mathbf{x}_n)$ .

- The Wronskian is  $W(t) = \det \Phi$ .
- The general solution is simply  $\Phi \mathbf{c}$ .
- $\Phi(t)$  satisfies the matrix equation  $\Phi' = A\Phi$ .

This is a consequence of how matrix multiplication works, and a good test of your understanding. Indeed,  $\Phi' = (\mathbf{x}'_1 \dots \mathbf{x}'_n)$  and  $A\Phi = A(\mathbf{x}_1 \dots \mathbf{x}_n) = (A\mathbf{x}_1 \dots A\mathbf{x}_n)$ .  $\diamond$

**Example 131.** Find a fundamental matrix for  $\mathbf{x}' = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}$ .

**Solution.** (using  $e^{At}$ ) Let  $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} = 2I + N$  with  $N = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ .

We want to use  $e^{At} = e^{2It+Nt} = e^{2It}e^{Nt}$ . This is indeed possible though we have to be careful:  $e^{A+B} = e^Ae^B$  holds if  $AB = BA$ . The identity matrix commutes with every other matrix, so we are good here.

$e^{2It}$  is simple to compute (so is the exponential of any diagonal matrix):  $e^{2It} = e^{2t}I$ .

$e^{Nt}$  is also simple to compute because  $N$  is nilpotent:  $N^2 = \begin{pmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

$$e^{Nt} = I + Nt + N^2 \frac{t^2}{2} + N^3 \frac{t^3}{6} + \dots = I + Nt + N^2 \frac{t^2}{2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -t & t \\ 0 & 0 & 3t \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{3}{2}t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -t & t - \frac{3}{2}t^2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{pmatrix}$$

Together,  $e^{At} = e^{2t} \begin{pmatrix} 1 & -t & t - \frac{3}{2}t^2 \\ 0 & 1 & 3t \\ 0 & 0 & 1 \end{pmatrix}$  is a fundamental matrix.

**Solution.** (using generalized eigenvectors) The characteristic polynomial is  $(2 - \lambda)^3$ . Hence,  $\lambda = 2$  is the only eigenvalue and has multiplicity 3. Solving the eigenvector equation  $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ , we (only) find  $\mathbf{v}_1 = (1, 0, 0)^T$  (or any multiple). This means that  $\lambda = 2$  has defect 2. There has to be a chain of length 3. To find a generalized eigenvector  $\mathbf{v}_2$  of rank 2, we solve  $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  and obtain, for instance,  $\mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ . Similarly, to find a generalized eigenvector  $\mathbf{v}_3$  of rank 3, we solve  $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$  and obtain, for instance,  $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1/3 \\ -1/3 \end{pmatrix}$ .

The corresponding solutions are  $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$ ,  $\mathbf{x}_2 = \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] e^{2t} = \begin{pmatrix} t \\ -1 \\ 0 \end{pmatrix} e^{2t}$  and  $\mathbf{x}_3 = \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/3 \\ -1/3 \end{pmatrix} \right] e^{2t} = \begin{pmatrix} t^2/2 \\ -1/3 - t \\ -1/3 \end{pmatrix} e^{2t}$ . Hence,  $\begin{pmatrix} 1 & t & t^2/2 \\ 0 & -1 & -1/3 - t \\ 0 & 0 & -1/3 \end{pmatrix} e^{2t}$  is a(nother) fundamental matrix.

The fundamental matrices look different but they are equivalent. Check it! (We did.)  $\diamond$

**Remark 132. (April Fools' Day!)**  $1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = ii = -1$ .

When using the principal square-root (which basically takes the positive root, that is, the one with positive real part), the rule  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  does not hold universally (so the trouble lies with the third equality). It does hold if, for instance,  $a \geq 0$  or  $b \geq 0$ . Apparently, this trouble historically resulted in lots of controversy around complex numbers, with some mathematicians rejecting them outright; though unjust, it has even been claimed<sup>24</sup> that Euler was confused about the law  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ .  $\diamond$

**Review.** Consider  $\mathbf{x}' = A\mathbf{x}$ , as usual.

- Any fundamental matrix  $\Phi(t)$  satisfies  $\Phi' = A\Phi$ .
- $e^{At}$  is a fundamental matrix. It is the unique fundamental matrix  $\Phi(t)$  with  $\Phi(0) = I$ .  $\diamond$

The matrix exponential has some nice properties:

- $e^{A+B} = e^A e^B$  if  $AB = BA$ .

This can be shown starting with the series  $e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$  by expanding  $(A+B)^n$  using the binomial theorem. For instance,  $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ . In order to simplify this to  $(A+B)^2 = A^2 + 2AB + B^2$ , we need  $AB = BA$ .

- $(e^A)^{-1} = e^{-A}$

This follows from the previous upon setting  $B = -A$ .

**Theorem 133.** If  $\Phi(t)$  is any fundamental matrix, then  $e^{At} = \Phi(t)\Phi(0)^{-1}$ . [see also Lecture 33]

**Proof.** Both sides are fundamental matrices (why is  $\Phi(t)\Phi(0)^{-1}$  another fundamental matrix?!) and, hence, satisfy the IVP  $\Phi' = A\Phi$ ,  $\Phi(0) = I$ . By uniqueness, they have to be equal.  $\square$

**Example 134.** Compute  $e^{At}$  for  $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ .

**Solution.** The characteristic polynomial is  $(2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$ .

Hence, the eigenvalues are  $\lambda = -1, 4$ .

$\lambda = -1$ :  $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , so  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

$\lambda = 4$ :  $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , so  $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ .

This gives us the fundamental matrix  $\Phi(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}$ .

$\Phi(0) = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ ,  $\Phi(0)^{-1} = \frac{1}{5} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$

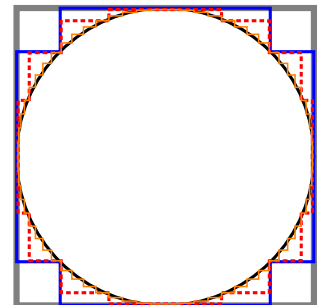
Hence,  $e^{At} = \Phi(t)\Phi(0)^{-1} = \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{4t} & -3e^{-t} + 3e^{4t} \\ -2e^{-t} + 2e^{4t} & 3e^{-t} + 2e^{4t} \end{pmatrix}$ .

Note that  $e^{At}|_{t=0} = I$ , indeed.  $\diamond$

**Theorem 135.** The IVP  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \mathbf{x}_0$  has solution  $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ .

**Proof.** The general solution is  $\mathbf{x}(t) = e^{At}\mathbf{c}$ .  $\mathbf{x}(0) = e^{At}\mathbf{c}|_{t=0} = \mathbf{c}$ . To solve the IVP, we choose  $\mathbf{c} = \mathbf{x}_0$ .  $\square$

**Remark 136. (April Fools' Day!)**  $\pi$  is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates  $\pi$ . To get a better approximation, we “fold” the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that  $\pi = 4$ , contrary to popular belief.  $\diamond$



24. See, for instance: [https://webspace.utexas.edu/aam829/1/m/Euler\\_files/EulerMonthly.pdf](https://webspace.utexas.edu/aam829/1/m/Euler_files/EulerMonthly.pdf)

## Inhomogeneous linear systems

$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$  is the general form of a (first-order<sup>25</sup>) inhomogeneous system of linear DEs.

- To solve it, we find a particular solution  $\mathbf{x}_p(t)$ .
- Then, the general solution is  $\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_c(t)$ .  
Here,  $\mathbf{x}_c(t)$  is the general solution of  $\mathbf{x}' = A\mathbf{x}$ .
- Two methods: undetermined coefficients and variation of constants

**Example 137.** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}$ .

**Solution.** From the previous lecture, we know that  $\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$ .

We look for a solution of the shape  $\mathbf{x}_p = \mathbf{a}e^{3t}$ . To determine the undetermined coefficients, we plug into the DE.

$$\mathbf{x}'_p = 3\mathbf{a}e^{3t} \stackrel{!}{=} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{a}e^{3t} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix} = \begin{pmatrix} 2a_1 + 3a_2 \\ 2a_1 + a_2 - 2 \end{pmatrix} e^{3t}$$

$$\text{Solving } \begin{pmatrix} 3a_1 \\ 3a_2 \end{pmatrix} = \begin{pmatrix} 2a_1 + 3a_2 \\ 2a_1 + a_2 - 2 \end{pmatrix}, \text{ we find } \mathbf{a} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}.$$

$$\text{Hence, the general solution is } \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} e^{3t} + c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}. \quad \diamond$$

**Example 138.** Find a particular solution of  $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}$ .

**Solution.** Duplication! The 4 (“new” root) from  $e^{4t}$  in the inhomogeneous part coincides with an eigenvalue (“old” root). In our particular solution, we therefore include a term  $\mathbf{a}te^{4t}$ . However, that is not enough; as in the case of generalized eigenvectors, we need to include lower order terms as well.

Hence, we look for a solution of the form  $\mathbf{x}_p = (\mathbf{a}t + \mathbf{b})e^{4t}$ . To determine  $\mathbf{a}$  and  $\mathbf{b}$ , we plug into the differential equation. (Note that  $\mathbf{b}$  will not be unique: if  $\mathbf{b}$  works, then so does any  $\mathbf{b} + c \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ . Why?!)

$$\mathbf{x}'_p = (4\mathbf{a}t + \mathbf{a} + 4\mathbf{b})e^{4t} \stackrel{!}{=} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} (\mathbf{a}t + \mathbf{b})e^{4t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t}$$

$$\text{Equating the coefficients of } te^{4t}, \text{ we get } 4\mathbf{a} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{a} \text{ or, equivalently, } \begin{pmatrix} 2a_1 - 3a_2 \\ -2a_1 + 3a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\text{Equating the coefficients of } e^{4t}, \text{ we get } \mathbf{a} + 4\mathbf{b} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{b} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ or, equivalently, } \begin{pmatrix} a_1 + 2b_1 - 3b_2 \\ a_2 - 2b_1 + 3b_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$\text{Adding the last two equations, we get } a_1 + a_2 = 3. \text{ Together with } 2a_1 - 3a_2 = 0, \text{ this gives } \mathbf{a} = \begin{pmatrix} 9/5 \\ 6/5 \end{pmatrix}.$$

$$\text{We are left with } 2b_1 - 3b_2 = 2 - a_1 = \frac{1}{5}. \text{ We choose } b_2 = 0, \text{ in which case we find } b_1 = \frac{1}{10}.$$

$$\text{In conclusion, we have found the particular solution } \mathbf{x}_p = \begin{pmatrix} 9/5t + 1/10 \\ 6/5t \end{pmatrix} e^{4t}. \quad \diamond$$

**Example 139.** Find a particular solution of  $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{4t} \\ e^{4t} - 2e^{3t} \end{pmatrix}$ .

$$\textbf{Solution.} \text{ Note that } \begin{pmatrix} 2e^{4t} \\ e^{4t} + e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}.$$

Using the particular solutions from the two previous examples, we therefore have the particular solution

$$\mathbf{x}_p = \begin{pmatrix} 9/5t + 1/10 \\ 6/5t \end{pmatrix} e^{4t} - \frac{1}{2} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} e^{3t}. \quad \diamond$$

<sup>25</sup>. Recall that any system can be written as a (larger) first-order system.

**Review.** undetermined coefficients ◇

**Example 140.** Consider  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ , where  $A$  is a  $7 \times 7$  matrix with eigenvalues  $1 \pm 2i$ ,  $1 \pm 2i$ ,  $0$ ,  $3$ ,  $3$ . For different choices of  $\mathbf{f}(t)$ , we set up  $\mathbf{x}_p$  with undetermined coefficients.

$\mathbf{f}(t)$	“new” roots	$\mathbf{x}_p$
$\mathbf{g}e^t$	1	$\mathbf{a}e^t$
$\mathbf{g}$	0	$\mathbf{a}t + \mathbf{b}$
$\mathbf{g} \sin(2t)$	$\pm 2i$	$\mathbf{a} \cos(2t) + \mathbf{b} \sin(2t)$
$\mathbf{g}e^t \sin(2t)$	$1 \pm 2i$	$(\mathbf{a}_1 t^2 + \mathbf{a}_2 t + \mathbf{a}_3)e^t \cos(2t) + (\mathbf{a}_4 t^2 + \mathbf{a}_5 t + \mathbf{a}_6)e^t \sin(2t)$
$\mathbf{g}(t^2 + 7)e^{3t}$	3, 3, 3	$(\mathbf{a}_1 t^4 + \mathbf{a}_2 t^3 + \mathbf{a}_3 t^2 + \mathbf{a}_4 t + \mathbf{a}_5)e^{3t}$
$\mathbf{g}(t^2 - 3t) + \mathbf{h}e^t \cos(t)$	0, 0, 0, $1 \pm i$	$(\mathbf{a}_1 t^3 + \mathbf{a}_2 t^2 + \mathbf{a}_3 t + \mathbf{a}_4) + \mathbf{a}_5 e^t \cos(t) + \mathbf{a}_6 e^t \sin(t)$

It should be remarked that, based on the information on  $A$  that we have, the forms for  $\mathbf{x}_p$  are for the “worst possible” case. If, for instance, the eigenvalue  $1 \pm 2i$  had no defect, then the form of  $\mathbf{x}_p$  for  $\mathbf{f}(t) = \mathbf{g}e^t \sin(2t)$  would simplify to  $\mathbf{x}_p = (\mathbf{a}_1 t + \mathbf{a}_2)e^t \cos(2t) + (\mathbf{a}_3 t + \mathbf{a}_4)e^t \sin(2t)$ . Do you see why? ◇

**Theorem 141.** (**variation of constants**) The DE  $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$  is solved by

$$\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) dt.$$

Here,  $\Phi(t)$  is any fundamental matrix for  $\mathbf{x}' = A(t)\mathbf{x}$ .

**Proof.** Recall that the general solution of the homogeneous equation  $\mathbf{x}' = A(t)\mathbf{x}$  is  $\mathbf{x}_c = \Phi(t)\mathbf{c}$ . We are going to vary the constant  $\mathbf{c}$  and look for a particular solution of the form  $\mathbf{x}_p = \Phi(t)\mathbf{u}(t)$ .

Plugging into the DE, we get

$$\mathbf{x}_p'(t) = \Phi'(t)\mathbf{u}(t) + \Phi(t)\mathbf{u}'(t) = A\Phi(t)\mathbf{u}(t) + \Phi(t)\mathbf{u}'(t) \stackrel{!}{=} A\mathbf{x}_p(t) + \mathbf{f}(t) = A\Phi(t)\mathbf{u}(t) + \mathbf{f}(t).$$

For the first equality, we used the matrix version of the usual product rule (which holds since differentiation is defined entry-wise). For the second equality, we used  $\Phi' = A\Phi$ .

Subtracting  $A\Phi\mathbf{u}$ , we see that  $\mathbf{x}_p = \Phi(t)\mathbf{u}(t)$  is a solution if and only if  $\Phi(t)\mathbf{u}'(t) = \mathbf{f}(t)$ .

Hence,  $\mathbf{u}'(t) = \Phi(t)^{-1} \mathbf{f}(t)$  and it only remains to integrate. □

**Example 142.** Find a particular solution of  $\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ -2e^{3t} \end{pmatrix}$ .

**Solution.** From previous examples, we know that  $\Phi(t) = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix}$ .

Using  $\det \Phi = 5e^{3t}$ , we find  $\Phi(t)^{-1} = \frac{1}{5} \begin{pmatrix} 2e^t & -3e^t \\ e^{-4t} & e^{-4t} \end{pmatrix}$ .

Hence,  $\Phi(t)^{-1} \mathbf{f}(t) = \frac{2}{5} \begin{pmatrix} 3e^{4t} \\ -e^{-t} \end{pmatrix}$  and  $\int \Phi(t)^{-1} \mathbf{f}(t) dt = \frac{2}{5} \begin{pmatrix} 3/4 e^{4t} \\ e^{-t} \end{pmatrix}$ .

By variation of constants,  $\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) dt = \begin{pmatrix} e^{-t} & 3e^{4t} \\ -e^{-t} & 2e^{4t} \end{pmatrix} \frac{2}{5} \begin{pmatrix} 3/4 e^{4t} \\ e^{-t} \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 15/4 \\ 5/4 \end{pmatrix} e^{3t} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} e^{3t}$ .

Note that this matches the result we obtained in Example 137.

By the way, why do we not need to be careful about the constants of integration? ◇



**Review.** Variation of constants:  $\mathbf{x}_p(t) = \Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) dt$  solves  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$

Here,  $\Phi(t)$  is any fundamental matrix of  $\mathbf{x}' = A\mathbf{x}$ . ◇

In the special case that  $\Phi(t) = e^{At}$ , some things become easier. For instance,  $\Phi(t)^{-1} = e^{-At}$ . Also, we can just write down solutions to IVPs:

- $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \mathbf{x}_0$  has (unique) solution  $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ .
- $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ ,  $\mathbf{x}(0) = \mathbf{x}_0$  has (unique) solution  $\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At} \int_0^t e^{-As} \mathbf{f}(s) ds$ .

**Example 143.** Suppose that the matrix  $A$  satisfies  $e^{At} = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}$ .

- Solve  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

**Solution.**  $\mathbf{x}(t) = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{pmatrix}$ .

- Solve  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 0 \\ 2e^t \end{pmatrix}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

**Solution.**  $\mathbf{x}(t) = e^{At} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{At} \int_0^t e^{-As} \mathbf{f}(s) ds$ . We compute:

$$\int_0^t e^{-As} \mathbf{f}(s) ds = \int_0^t \begin{pmatrix} 2e^{-2s} - e^{-3s} & -2e^{-2s} + 2e^{-3s} \\ e^{-2s} - e^{-3s} & -e^{-2s} + 2e^{-3s} \end{pmatrix} \begin{pmatrix} 0 \\ 2e^s \end{pmatrix} ds = \int_0^t \begin{pmatrix} -4e^{-s} + 4e^{-2s} \\ -2e^{-s} + 4e^{-2s} \end{pmatrix} ds = \begin{pmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{pmatrix}$$

$$\text{Hence, } e^{At} \int_0^t e^{-As} \mathbf{f}(s) ds = \begin{pmatrix} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix} \begin{pmatrix} 4e^{-t} - 2e^{-2t} - 2 \\ 2e^{-t} - 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{pmatrix}.$$

$$\text{Finally, } \mathbf{x}(t) = \begin{pmatrix} -2e^{2t} + 3e^{3t} \\ -e^{2t} + 3e^{3t} \end{pmatrix} + \begin{pmatrix} 2e^t - 4e^{2t} + 2e^{3t} \\ -2e^{2t} + 2e^{3t} \end{pmatrix} = \begin{pmatrix} 2e^t - 6e^{2t} + 5e^{3t} \\ -3e^{2t} + 5e^{3t} \end{pmatrix}.$$

- What is  $A$ ?

**Solution.** Like any fundamental matrix,  $e^{At}$  satisfies  $\frac{d}{dt}e^{At} = Ae^{At}$ .

$$\text{Hence, } A = \left[ \frac{d}{dt}e^{At} \right]_{t=0} = \left[ \begin{pmatrix} 4e^{2t} - 3e^{3t} & -4e^{2t} + 6e^{3t} \\ 2e^{2t} - 3e^{3t} & -2e^{2t} + 6e^{3t} \end{pmatrix} \right]_{t=0} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \quad \diamond$$

**Example 144.** Three brine tanks  $T_1, T_2, T_3$ .

This is Example 5.6.2 in the book.

$T_1$  contains 20gal water with 10lb salt,  $T_2$  40gal pure water,  $T_3$  50gal water with 30lb salt.

$T_1$  is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of  $T_1$  into  $T_2$ . Also, 10gal/min well-mixed solution flows out of  $T_2$  into  $T_3$ . Finally, 10gal/min well-mixed solution is leaving  $T_3$ . How much salt is in the tanks after  $t$  minutes?

**Solution.** Let  $x_i(t)$  denote the amount of salt (in lb) in tank  $T_i$  after time  $t$  (in min).

In time interval  $[t, t + \Delta t]$ :

$$\Delta x_1 \approx 10 \cdot 2 \cdot \Delta t - 10 \cdot \frac{x_1}{20} \cdot \Delta t, \text{ so } x'_1 = 20 - \frac{1}{2}x_1. \text{ Also, } x_1(0) = 10.$$

$$\Delta x_2 \approx 10 \cdot \frac{x_1}{20} \cdot \Delta t - 10 \cdot \frac{x_2}{40} \cdot \Delta t, \text{ so } x'_2 = \frac{1}{2}x_1 - \frac{1}{4}x_2. \text{ Also, } x_2(0) = 0.$$

$$\Delta x_3 \approx 10 \cdot \frac{x_2}{40} \cdot \Delta t - 10 \cdot \frac{x_3}{50} \cdot \Delta t, \text{ so } x'_3 = \frac{1}{4}x_2 - \frac{1}{5}x_3. \text{ Also, } x_3(0) = 30.$$

$$\text{Using matrix notation and writing } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ this is } \mathbf{x}' = \begin{pmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x}(0) = \begin{pmatrix} 10 \\ 0 \\ 30 \end{pmatrix}.$$

We can solve this IVP! (Details in our book.)

Here, we content ourselves with a particular solution (and ignoring the initial conditions). Undetermined coefficients tells us that there is a solution of the form  $\mathbf{x}_p(t) = \mathbf{a}$ . Of course, we can find  $\mathbf{a}$  by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find  $\mathbf{x}_p = (40, 80, 100)$  without calculation. ◇



## Fourier series

**Definition 145.** Let  $L > 0$ .  $f(t)$  is  $L$ -periodic if  $f(t + L) = f(t)$  for all  $t$ . The smallest such  $L$  is called “the” period of  $f$ .

**Example 146.** The period of  $\cos(nt)$  is  $2\pi/n$ . ◇

**Example 147.** The trigonometric functions  $\cos(nt)$ ,  $\sin(nt)$  are  $2\pi$ -periodic. And so are all their linear combinations. (In other words,  $2\pi$ -periodic functions form a vector space.) ◇

The following amazing fact is saying that any  $2\pi$ -periodic function can be written as a sum of cosines and sines.

**Theorem 148.** Every\*  $2\pi$ -periodic function  $f$  can be written as a Fourier series<sup>26</sup>

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Technical detail\*:  $f$  needs to be, e.g., piecewise smooth.

Also, if  $t$  is a discontinuity, then the Fourier series converges to the average  $\frac{f(t^-) + f(t^+)}{2}$ .

The Fourier coefficients  $a_n$ ,  $b_n$  are unique and can be computed as

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt.$$

**Example 149.** Find the Fourier series of the  $2\pi$ -periodic function  $f(t)$  defined by

$$f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi. \end{cases}$$

**Solution.** We compute  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = 0$ , as well as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \frac{1}{\pi} \left[ - \int_{-\pi}^0 \cos(nt) dt + \int_0^{\pi} \cos(nt) dt \right] = 0 \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt = \frac{1}{\pi} \left[ - \int_{-\pi}^0 \sin(nt) dt + \int_0^{\pi} \sin(nt) dt \right] = \frac{2}{\pi n} [1 - \cos(n\pi)] \\ &= \frac{2}{\pi n} [1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \end{aligned}$$

In conclusion,  $f(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(nt) = \frac{4}{\pi} \left( \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \dots \right)$ . ◇

**Remark 150. (JustForFun)** Set  $t = \frac{\pi}{2}$  in the Fourier series we just computed, to get Leibniz' series<sup>27</sup>  $\pi = 4 \left[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$ ; which brings us back to our April fool's derivation of  $\pi = 4$ . The trouble boils down to the fact that we would like to conclude that the convergence of functions  $f_n \rightarrow f$  implies that their arc length  $L(f_n)$  converges to  $L(f)$ . That's a natural instinct (and would be true if  $L$  is continuous). However, arc length depends on the derivative (remember its formula?!), and  $f_n \rightarrow f$  does not<sup>28</sup> necessarily imply  $f'_n \rightarrow f'$ . ◇

26. Another common way to write Fourier series is  $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ . These two ways are equivalent; we can convert between them using Euler's identity  $e^{int} = \cos(nt) + i \sin(nt)$ .

27. For such an alternating series, the error made by stopping at the term  $1/n$  is on the order of  $1/n$ . To compute the 768 digits of  $\pi$  to get to the Feynman point, we would (roughly) need  $1/n < 10^{-768}$ , or  $n > 10^{768}$ . That's a lot of terms! (Roger Penrose, for instance, estimates that there are about  $10^{80}$  atoms in the observable universe.)

28. In other words, taking the derivative of a function is not a continuous operator! That's a subject for functional analysis, which studies spaces of functions as well as operators between such spaces.

There was nothing special about  $2\pi$ -periodic functions considered last time (except that  $\cos(t)$  and  $\sin(t)$  have period  $2\pi$ ). Note that  $\cos(\pi t/L)$  has period  $2L$ .

**Theorem 151.** Every\*  $2L$ -periodic function  $f$  can be written as a **Fourier series**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right).$$

Technical detail\*:  $f$  needs to be, e.g., piecewise smooth.

Also, if  $t$  is a discontinuity, then the Fourier series converges to the average  $\frac{f(t^-) + f(t^+)}{2}$ .

The **Fourier coefficients**  $a_n$ ,  $b_n$  are unique and can be computed as

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt.$$

**Review.** Last time, we computed  $f(t) = \begin{cases} -1, & \text{for } t \in (-\pi, 0), \\ +1, & \text{for } t \in (0, \pi), \\ 0, & \text{for } t = -\pi, 0, \pi \end{cases} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t).$  ◇

**Example 152.** Find the Fourier series of the 2-periodic function  $g(t) = \begin{cases} -1 & \text{for } t \in (-1, 0) \\ +1 & \text{for } t \in (0, 1) \\ 0 & \text{for } t = -1, 0, 1 \end{cases}.$

**Solution.** Instead of computing from scratch, we can use the fact that  $g(t) = f(\pi t)$ , with  $f$  as reviewed above, to get  $g(t) = f(\pi t) = \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(n\pi t).$  ◇

**Remark 153.** Convergence of such series is not obvious! Recall, for instance, that the (odd part of) the harmonic series  $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$  diverges. ◇

**Theorem 154.** If  $f(t)$  is **continuous** and  $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right)$ , then\*  $f'(t) = \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} b_n \cos\left(\frac{n\pi t}{L}\right) - \frac{n\pi}{L} a_n \sin\left(\frac{n\pi t}{L}\right) \right)$  (i.e., we can differentiate termwise).

Technical detail\*:  $f'$  needs to be, e.g., piecewise smooth (so that it has a Fourier series itself).

**Example 155.** Let  $h(t)$  be the 2-periodic function with  $h(t) = \begin{cases} -t & \text{for } t \in (-1, 0) \\ +t & \text{for } t \in (0, 1) \end{cases}$ . Compute the Fourier series of  $h(t)$ .

**Solution.** We could just use the integral formulas to compute  $a_n$  and  $b_n$ . Since  $h(t)$  is even (plot it!), we will find that  $b_n = 0$ . Computing  $a_n$  is left as an exercise.

**Solution.** Note that  $h(t)$  is continuous and  $h'(t) = g(t)$ , with  $g(t)$  as in Example 152. Hence, we can apply Theorem 154 to conclude

$$h'(t) = g(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi t) \implies h(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \left( -\frac{1}{\pi n} \right) \cos(n\pi t) + C,$$

where  $C = \frac{a_0}{2} = \frac{1}{2} \int_{-1}^1 h(t) dt$  is the constant of integration. Thus,  $h(t) = \frac{1}{2} - \sum_{n \text{ odd}} \frac{4}{\pi^2 n^2} \cos(n\pi t).$  ◇

**Remark 156.** Note that  $t=0$  in the last Fourier series, gives us  $\frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ . As an exercise, you can try to find from here the fact that  $\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$ . Similarly, we can use Fourier series to find that  $\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$ . JFF: if you recall from lecture 13, these are the values  $\zeta(2)$  and  $\zeta(4)$  of the Riemann zeta function  $\zeta(s)$ . No such values are known for  $\zeta(3), \zeta(5), \dots$ . Nobody believes these to be rational numbers, but it was only in 1978 that Apéry proved that  $\zeta(3)$  is not a rational number<sup>29</sup>. ◇

**Example 157.** The function  $g(t)$ , from in Example 152, is not continuous. For all values, except the discontinuities, we have  $g'(t) = 0$ . On the other hand, differentiating the Fourier series termwise, results in  $4 \sum_{n \text{ odd}} \cos(n\pi t)$ , which diverges<sup>30</sup> for most values of  $t$  (that's easy to check for  $t=0$ ). This illustrates that we cannot apply Theorem 154 because of the missing continuity. ◇

29. We also know that at least one of  $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$  is not a rational number. (Our state of ignorance!)

30. The issues we are facing here can be fixed by generalizing the notion of function to distributions. (Maybe you have heard of the Dirac delta "function".)

Review. Fourier series



## Fourier series and differential equations

Let us revisit the inhomogeneous equation  $mx'' + kx = F(t)$  describing the motion of a mass  $m$  on a spring with spring constant  $k$  under the influence of an external force  $F(t)$ .

Recall that, when  $F = 0$  (the complementary homogeneous equation), then the solutions are combinations of  $\cos(\omega_0 t)$  and  $\sin(\omega_0 t)$ , where  $\omega_0 = \sqrt{k/m}$  is the [natural frequency](#).

We have solved equations like  $mx'' + kx = \sin(\omega t)$ . A crucial insight was that the case  $\omega = \omega_0$  (overlapping roots) is special and corresponds to [resonance](#).

We are now going to allow any periodic force  $F(t)$ , and solve the equation by using the Fourier series for  $F(t)$ . The same approach works likewise for linear equations of higher order, or even systems of equations.

**Example 158.** Find a particular solution of  $2x'' + 32x = F(t)$ , with  $F(t) = \begin{cases} 10 & \text{if } t \in (0, 1) \\ -10 & \text{if } t \in (1, 2) \end{cases}$ , extended 2-periodically.

**Solution.** Step A: From the previous classes, we already know  $F(t) = 10 \sum_{n \text{ odd}} \frac{4}{\pi n} \sin(\pi n t)$ .

Step B: We next solve the equation  $2x'' + 32x = \sin(\pi n t)$  for  $n = 1, 3, 5, \dots$ . First, we note that the external frequency is  $\pi n$ , which is never equal to the natural frequency  $\omega_0 = 4$ . Hence, there exists a particular solution of the form  $x_p = A \cos(\pi n t) + B \sin(\pi n t)$ . To determine the coefficients  $A, B$ , we plug into the DE. Noting that  $x_p'' = -\pi^2 n^2 x_p$  (why?!), we get

$$2x_p'' + 32x_p = (32 - 2\pi^2 n^2)(A \cos(\pi n t) + B \sin(\pi n t)) \stackrel{!}{=} \sin(\pi n t).$$

We conclude  $A = 0$  and  $B = \frac{1}{32 - 2\pi^2 n^2}$ , so that  $x_p = \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}$ .

Step C: We combine the particular solutions found in the previous step, to see that

$$2x'' + 32x = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(\pi n t) \quad \text{is solved by} \quad x_p = 10 \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{\sin(\pi n t)}{32 - 2\pi^2 n^2}.$$

Note that  $x_p(t) = 1.038 \sin(\pi t) - 0.029 \sin(3\pi t) - 0.0055 \sin(5\pi t) - \dots$  is well approximated by the first two terms. Indeed, the amplitude of  $x_p$  is about  $1.038 + 0.029$  [first two terms have a maximum at  $t = 1/2$ ].

**Example 159.** Find a particular solution of  $2x'' + 32x = F(t)$ , with  $F(t)$  the  $2\pi$ -periodic function such that  $F(t) = 10t$  for  $t \in (-\pi, \pi)$ .

**Solution.** Step A: The Fourier series of  $F(t)$  is  $F(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt)$ . [Exercise!]

Step B: We next solve the equation  $2x'' + 32x = \sin(nt)$  for  $n = 1, 2, 3, \dots$ . Note, however, that [\(pure\) resonance](#) does occur for  $n = 4$ , so we need to treat that case separately. If  $n \neq 4$  then we find, as in the previous example, that  $x_p = \frac{\sin(nt)}{32 - 2n^2}$ . [See how this fails for  $n = 4$ !]

For  $2x'' + 32x = \sin(4t)$ , we begin with  $x_p = At \cos(4t) + Bt \sin(4t)$ . Then  $x_p' = (A + 4Bt)\cos(4t) + (B - 4At)\sin(4t)$ , and  $x_p'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$ . Plugging into the DE, we get  $2x_p'' + 32x_p = 16B \cos(4t) - 16A \sin(4t) \stackrel{!}{=} \sin(4t)$ , and thus  $B = 0$ ,  $A = -\frac{1}{16}$ . So,  $x_p = -\frac{1}{16}t \cos(4t)$ .

Step C: We combine the particular solutions to get

$$2x'' + 32x = -5 \sin(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \sin(nt) \quad \text{is solved by} \quad x_p = \frac{5}{16}t \cos(4t) + \sum_{\substack{n=1 \\ n \neq 4}}^{\infty} (-1)^{n+1} \frac{20}{n} \frac{\sin(nt)}{32 - 2n^2}.$$

As in the previous example, this solution cannot really be simplified. Make some plots to appreciate the dominating character of the term resulting from resonance!

## Inverting matrices of any size

In order to compute  $A^{-1}$ , we need to find a matrix  $X$  such that  $AX = I$ . If this equation has a solution  $X$ , then  $X = A^{-1}$ . As we have done before, we write  $A|I$  and perform elimination on the rows. Instead of stopping at a triangular shape, we continue until we get  $I|B$ . Then  $A^{-1} = B$ . This is best explained by an example (which we can already do).

**Example 160.** Let  $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ . Find  $A^{-1}$ .

**Solution.** We eliminate

$$\begin{array}{c|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array} \xrightarrow[r_2 - \frac{2}{3}r_1]{\frac{1}{3}r_1} \begin{array}{c|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 1 \end{array} \xrightarrow[3r_2]{r_1 - r_2} \begin{array}{c|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array} \Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}.$$

**Solution.** Using  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ , we again find  $A^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ .  $\diamond$

**Example 161.** Find  $e^{At}$  if  $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$ .

**Solution.** We first compute a fundamental matrix  $\Phi(t)$  for  $\mathbf{x}' = A\mathbf{x}$ . To begin with, we easily see that the eigenvalues are  $\lambda = 1, 1, 2$  (why?!).

$\lambda = 2$ .  $\begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{v} = 0$ . We find the eigenvector  $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

$\lambda = 1$ .  $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{v} = 0$ . We find the eigenvector  $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  but no second independent eigenvector.  $\lambda = 1$

has defect 1. We therefore solve  $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ , and find, for instance,  $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ .

Taken together, we have found that  $\Phi(t) = \begin{pmatrix} e^t & te^t & 0 \\ 0 & \frac{1}{2}e^t & 0 \\ 0 & \frac{1}{2}e^t & e^{2t} \end{pmatrix}$  is a fundamental matrix.

We can now find  $e^{At}$  from  $e^{At} = \Phi(t)\Phi(0)^{-1}$ .

$$\Phi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}, \begin{array}{c|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1 & 0 & 0 & 1 \end{array} \xrightarrow[r_3 - r_2]{2r_2} \begin{array}{c|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}, \quad \Phi(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Hence,

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} e^t & te^t & 0 \\ 0 & \frac{1}{2}e^t & 0 \\ 0 & \frac{1}{2}e^t & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 2te^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t - e^{2t} & e^{2t} \end{pmatrix}.$$

**Solution. (failed attempt)** We can write<sup>31</sup>  $A = D + N$  with  $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  and  $N = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ . We quickly check that  $N$  is nilpotent: in fact,  $N^2 = 0$ . Therefore,

$$e^{Dt}e^{Nt} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} (I + Nt) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 2t & 0 \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{pmatrix} = \begin{pmatrix} e^t & 2te^t & 0 \\ 0 & e^t & 0 \\ 0 & -te^{2t} & e^{2t} \end{pmatrix}.$$

This cannot be  $e^{At}$  because of the entry  $-te^{2t}$ . What went wrong?! Well, in order to use  $e^{At} = e^{Dt}e^{Nt}$  we first need to check that  $DN = ND$ . This is not the case here!  $\diamond$

31. Here is a twist on our usual approach, which can be used here: write  $A = I + B$ . Then  $B$  is not nilpotent, but we observe that  $B^2 = B^3 = B^4 = \dots$ . Do you see how to use this to compute  $e^{Bt}$ , and then  $e^{At}$ ?

## Review: basic skills

We have learned quite a bit about complex numbers and linear algebra. These are also very useful for your general (mathematical) well-being outside of DEs. Here is a rough overview of what we got to know.

- We can calculate with (e.g. divide) complex numbers. Real and imaginary part.
- We are still amazed by Euler's identity  $e^{i\theta} = \cos\theta + i\sin\theta$ .
- Add and multiply vectors and matrices. Identity matrix.
- Compute **determinants** of matrices by row (or column, if you wish) expansion.  
The determinant is zero  $\iff$  the columns (or, equivalently, rows) are linearly dependent.
- Invert matrices (at least  $2 \times 2$ ).
- Find **eigenvalues**  $\lambda$  of a matrix. These are the roots of the **characteristic polynomial**  $\det(A - \lambda I)$ .  
If the matrix is  $n \times n$ , then the characteristic polynomial is of degree  $n$ . Over the complex numbers there are always  $n$  roots/eigenvalues if we count with repetition.
- For each eigenvalue there is at least one **eigenvector**  $\mathbf{v}$  and we know how to find it. If  $\lambda$  is a repeated, say  $m$  times, we may find up to  $m$  independent eigenvectors. If we find less, say only  $k < m$ , then  $\lambda$  is said to have **defect**  $m - k$ .
- If  $\lambda$  is defective, then we know that we can find **generalized eigenvectors**. These come in chains.
- We know how to take the exponential of a matrix:  $e^A$

How was  $e^A$  defined? Well, there is options... what is your favourite definition of  $e^a$  when  $a$  is just a number?

Definition via Taylor series:  $e^a = 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \dots$  works just as well for matrices  $e^A = 1 + A + \frac{A^2}{2} + \frac{A^3}{6} + \dots$

Via derivative:  $e^{at}$  is unique  $x(t)$  such that  $x' = ax$ ,  $x(0) = 1$  vs.  $e^{At}$  is unique  $\Phi(t)$  such that  $\Phi' = A\Phi$ ,  $\Phi(0) = I$

## Review: systems of DEs

We spent basically all the time since the last midterm on systems of DEs. Here is a reminder why and where we got.

- Any high-order DE can be transformed into a **first-order system**.  
That's why we have been studying  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$  for so long. It is not some esoteric special case that happens to be doable—far from that: any linear DE can be written in this form!! [And any DE can be approximated by a linear DE.]
- For linear systems  $\mathbf{x}' = A(t)\mathbf{x}$  existence and uniqueness of solutions is for free.  
... on the interval  $I$  where the entries of  $A(t)$  are continuous.
- We are familiar with the **Wronskian** and **fundamental matrices**.  
The matrix exponential  $e^{At}$  is a particularly nice fundamental matrix. If  $\Phi(t)$  is some fundamental matrix, then  $e^{At} = \Phi(t)\Phi(0)^{-1}$ .
- We can solve all homogeneous equations  $\mathbf{x}' = A\mathbf{x}$  where  $A$  has constant entries.
  - First, find eigenvalues  $\lambda$ . For each  $\lambda$ , we then determine the eigenvectors. If  $\lambda$  turns out to be defective, then we have to look for generalized eigenvectors.
  - Here's a reminder how to get solutions out of a chain  $\mathbf{v}_1, \dots, \mathbf{v}_k$  of generalized eigenvectors for  $\lambda$ :
 
$$\begin{aligned} (A - \lambda I)\mathbf{v}_1 &= 0 && \text{solution: } \mathbf{v}_1 e^{\lambda t} \\ (A - \lambda I)\mathbf{v}_2 &= \mathbf{v}_1 && \text{solution: } (\mathbf{v}_1 t + \mathbf{v}_2) e^{\lambda t} \\ &\vdots && \\ (A - \lambda I)\mathbf{v}_k &= \mathbf{v}_{k-1} && \text{solution: } \left( \mathbf{v}_1 \frac{t^{k-1}}{(k-1)!} + \mathbf{v}_2 \frac{t^{k-2}}{(k-2)!} + \dots + \mathbf{v}_{k-1} t + \mathbf{v}_k \right) e^{\lambda t}. \end{aligned}$$
  - If  $\lambda = a + bi$  is a complex eigenvalue, then it occurs together with its conjugate  $a - bi$ . We can get real-valued solutions by taking real and imaginary part of the complex solutions.  
We only need to do that for one of  $a \pm bi$  because the other will give rise to equivalent solutions.
- We learned how to solve **inhomogeneous equations**  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ .
  - If  $\mathbf{x}_p(t)$  is some particular solution, then  $\mathbf{x}_p(t) + \mathbf{x}_c(t)$  is the general solution.  
Here,  $\mathbf{x}_c(t)$  denotes the general solution of the complementary equation  $\mathbf{x}' = A\mathbf{x}$ .
  - We know two methods to find an  $\mathbf{x}_p(t)$ : **undetermined coefficients** and **variation of constants**. Variation of constants, that is  $\Phi(t) \int \Phi(t)^{-1} \mathbf{f}(t) dt$ , can always to be used, whereas undetermined coefficients requires  $\mathbf{f}(t)$  to be a linear combination of polynomials times exponentials (so that we can attach a "root" to it).

## The Laplace transform

**Definition 162.** The Laplace transform of a function  $f(t)$ ,  $t \geq 0$ , is defined as the new function

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

We also write  $\mathcal{L}(f(t)) = F(s)$ .

Note that, in order for the integral to exist,  $f(t)$  should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions, we are interested in (and we will not dwell on this issue).

$f(t)$	$F(s)$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$e^{at}$	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$

**Example 163.**

$$\mathcal{L}(e^{at}) = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{(a-s)t} dt = \left[ \frac{1}{a-s} e^{(a-s)t} \right]_{t=0}^{\infty} = 0 - \frac{1}{a-s} = \frac{1}{s-a}$$

Note that we needed  $a - s < 0$  in order for the integral to converge. Hence the Laplace transform has domain  $s > a$ . (During this introduction, we will not care too much about these technical details.)  $\diamond$

**Example 164.** The Laplace transform is linear:

$$\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = \int_0^{\infty} e^{-st} (c_1 f_1(t) + c_2 f_2(t)) dt = c_1 \int_0^{\infty} e^{-st} f_1(t) dt + c_2 \int_0^{\infty} e^{-st} f_2(t) dt,$$

which equals  $c_1 F_1(s) + c_2 F_2(s)$ .  $\diamond$

**Example 165.** By Euler's identity,  $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ . Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i \mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2 + \omega^2} + i \frac{\omega}{s^2 + \omega^2}.$$

Matching real and imaginary parts, gives  $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$  and  $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$ .  $\diamond$

**Example 166.** Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^{\infty} e^{-st} f'(t) dt = [e^{-st} f(t)]_{t=0}^{\infty} + \int_0^{\infty} s e^{-st} f(t) dt = sF(s) - f(0).$$

In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s \mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2 F(s) - sf(0) - f'(0). \quad \diamond$$

**Example 167.** Consider the (very simple) IVP  $x'(t) - 2x(t) = 0$ ,  $x(0) = 7$ .

[Of course,  $x(t) = 7e^{2t}$ .]

$$\mathcal{L}(x'(t) - 2x(t)) = \mathcal{L}(x'(t)) - 2\mathcal{L}(x(t)) = sX(s) - x(0) - 2X(s) = (s-2)X(s) - 7 = 0.$$

This is an algebraic equation for  $X(s)$ . It follows that  $X(s) = \frac{7}{s-2}$ . By inverting the Laplace transform (which is possible!), we conclude that  $x(t) = 7e^{2t}$ .  $\diamond$

**Review.** Laplace transform, table from last class

**Example 168.**  $\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a)$

**Example 169.** We also add the following to our table of Laplace transforms.

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)dt = \int_0^\infty -\frac{d}{ds}e^{-st}f(t)dt = -\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = -F'(s)$$

In particular,

$$\begin{aligned}\mathcal{L}(t) &= \mathcal{L}(t \cdot 1) = -\frac{d}{ds} \frac{1}{s} = \frac{1}{s^2} \\ \mathcal{L}(t^2) &= -\frac{d}{ds} \frac{1}{s^2} = \frac{2}{s^3} \\ &\vdots \\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.\end{aligned}$$

**Theorem 170. (Uniqueness of Laplace transforms)** If  $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$ , then  $f_1(t) = f_2(t)$ .

At least for all  $t$ , for which  $f_1(t)$  and  $f_2(t)$  are continuous. (Note that redefining  $f(t)$  at a single point, will not change its Laplace transform.)

Hence, we can recover  $f(t)$  from  $F(s)$ . We write  $\mathcal{L}^{-1}(F(s)) = f(t)$ .

**Example 171.** If  $F(s) = \frac{3s-7}{s^2+4}$ , what is  $f(t)$ ?

**Solution.**  $F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$ . Hence,  $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$ .

**Example 172.** If  $F(s) = \frac{1}{(s-3)^2}$ , what is  $f(t)$ ?

**Solution.**  $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$ .

**Example 173.** Solve the IVP  $x'' - 3x' + 2x = e^{-t}$ ,  $x(0) = 0$ ,  $x'(0) = 1$ .

**Solution. (old style)** The characteristic polynomial is  $s^2 - 3s + 2 = (s-1)(s-2)$ . Since there is no duplication, there is a particular solution  $x_p = ae^{-t}$ . To determine  $a$ , we compute  $x_p'' - 3x_p' + 2x_p = 6ae^{-t} \stackrel{!}{=} e^{-t}$  and conclude  $a = \frac{1}{6}$ . The general solution thus is  $x(t) = \frac{1}{6}e^{-t} + c_1e^t + c_2e^{2t}$ . Solving  $x(0) = \frac{1}{6} + c_1 + c_2 \stackrel{!}{=} 0$  and  $x'(0) = -\frac{1}{6} + c_1 + 2c_2 \stackrel{!}{=} 1$ , we find  $c_2 = \frac{4}{3}$  and  $c_1 = -\frac{3}{2}$ . Hence,  $x(t) = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$ .

**Solution. (Laplace style)**

$$\begin{aligned}\mathcal{L}(x''(t)) - 3\mathcal{L}(x'(t)) + 2\mathcal{L}(x(t)) &= \mathcal{L}(e^{-t}) \\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= \frac{1}{s+1} \\ (s^2 - 3s + 2)X(s) &= 1 + \frac{1}{s+1} = \frac{s+2}{s+1} \\ X(s) &= \frac{s+2}{(s-1)(s-2)(s+1)}\end{aligned}$$

To find  $x(t)$ , we use [partial fractions](#) to write  $X(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$ . We find the coefficients as

$$A = \frac{s+2}{(s-2)(s+1)} \Big|_{s=1} = -\frac{3}{2}, \quad A = \frac{s+2}{(s-1)(s+1)} \Big|_{s=2} = \frac{4}{3}, \quad C = \frac{s+2}{(s-1)(s-2)} \Big|_{s=-1} = \frac{1}{6}.$$

Finally,  $x(t) = \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}\right) = Ae^t + Be^{2t} + Ce^{-t} = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$ , as above.



**Review.** Laplace transform, go over last example ◇

**Example 174.** Solve the IVP  $x'' - 3x' + 2x = e^t$ ,  $x(0) = 0$ ,  $x'(0) = 1$ .

**Solution. (old style, outline)** The characteristic polynomial is  $s^2 - 3s + 2 = (s - 1)(s - 2)$ . Since there is duplication, we have to look for a particular solution of the form  $x_p = ate^t$ . To determine  $a$ , we need to plug into the DE (we find  $a = -1$ ). Then, the general solution is  $x(t) = ate^t + c_1e^t + c_2e^{2t}$ , and the initial conditions determine  $c_1$  and  $c_2$  (we find  $c_1 = -2$  and  $c_2 = 2$ ).

**Solution. (Laplace style)**

$$\begin{aligned} \mathcal{L}(x''(t)) - 3\mathcal{L}(x'(t)) + 2\mathcal{L}(x(t)) &= \mathcal{L}(e^t) \\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= \frac{1}{s-1} \\ (s^2 - 3s + 2)X(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1} \\ X(s) &= \frac{s}{(s-1)^2(s-2)} \end{aligned}$$

To find  $x(t)$ , we again use partial fractions.  $X(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$  with coefficients (why?!)

$$C = \frac{s}{(s-1)^2} \Big|_{s=2} = 2, \quad A = \frac{s}{s-2} \Big|_{s=1} = -1, \quad B = \frac{d}{ds} \frac{s}{s-2} \Big|_{s=1} = \frac{-2}{(s-2)^2} \Big|_{s=1} = -2.$$

Finally,  $x(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$ . ◇

**Example 175.** Solve the IVP  $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix} \mathbf{x}$ ,  $\mathbf{x} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ .

**Solution. (old style)** See Problem 2 on our practice problems for the final midterm exam. There we computed that

$$e^{At} = \begin{pmatrix} 3 - 2e^t & -1 + e^t \\ 6 - 6e^t & -2 + 3e^t \end{pmatrix}.$$

Hence,  $\mathbf{x} = e^{At} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 + 2e^t \\ -2 + 6e^t \end{pmatrix}$ .

**Solution. (Laplace style)** Writing  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ , the IVP is equivalent to

$$\begin{aligned} x_1' &= -2x_1 + x_2, & x_1(0) &= 1, \\ x_2' &= -6x_1 + 3x_2, & x_2(0) &= 4. \end{aligned}$$

Taking the Laplace transform of both equations, we get

$$\begin{aligned} sX_1(s) - x_1(0) &= -2X_1(s) + X_2(s), \\ sX_2(s) - x_2(0) &= -6X_1(s) + 3X_2(s), \end{aligned}$$

or, equivalently,

$$\begin{aligned} (s+2)X_1(s) - X_2(s) &= 1, \\ 6X_1(s) + (s-3)X_2(s) &= 4. \end{aligned}$$

Adding  $(s-3)$  times the first equation to the second one, we find  $(6 + (s-3)(s+2))X_1(s) = s(s-1)X_1(s) = s+1$ . Hence,

$$X_1(s) = \frac{s+1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}, \quad A = \frac{s+1}{s-1} \Big|_{s=0} = -1, \quad B = \frac{s+1}{s} \Big|_{s=1} = 2.$$

Taking the inverse Laplace transform, it follows that  $x_1(t) = -1 + 2e^t$ . Similarly,

$$X_2(s) = (s+2)X_1(s) - 1 = \frac{(s+2)(s+1)}{s(s-1)} - 1 = \frac{4s+2}{s(s-1)} = \frac{C}{s} + \frac{D}{s-1}, \quad C = \frac{4s+2}{s-1} \Big|_{s=0} = -2, \quad D = \frac{4s+2}{s} \Big|_{s=1} = 6,$$

which implies  $x_2(t) = -2 + 6e^t$ . In conclusion,  $\mathbf{x} = \begin{pmatrix} -1 + 2e^t \\ -2 + 6e^t \end{pmatrix}$ , as above. ◇



Let  $u_a(t) = \begin{cases} 1, & \text{if } t \geq a, \\ 0, & \text{if } t < a, \end{cases}$  be the **unit step function**<sup>32</sup>.

**Example 176.**  $\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t) dt = \int_a^\infty e^{-st} dt = \left[ -\frac{e^{-st}}{s} \right]_{t=a}^\infty = \frac{e^{-sa}}{s}.$   $\diamond$

**Example 177.** Note that  $u_a(t)f(t-a)$  is  $f(t)$  delayed by  $a$  (make a sketch!). We find

$$\mathcal{L}(u_a(t)f(t-a)) = \int_a^\infty e^{-st} f(t-a) dt = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) d\tilde{t} = e^{-sa} F(s). \quad \diamond$$

**Example 178.** What is  $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right)$ ?

**Solution.**  $\frac{1}{s+1}$  is the Laplace transform of  $e^{-t}$ . Hence,  $\frac{e^{-2s}}{s+1}$  is the Laplace transform of  $e^{-t}$  delayed by 2. In other words,  $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) = u_2(t)e^{-(t-2)}.$   $\diamond$

The next example illustrates that any piecewise defined function can be written using a single equation involving step functions. This is based on the simple observation that  $u_a(t) - u_b(t)$  is a function which is 1 on the interval  $[a, b)$  but zero everywhere else.

**Example 179.** Consider  $f(t) = \begin{cases} t^2, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } 1 \leq t \leq 2, \\ \cos(t-2), & \text{if } t \geq 2. \end{cases}$

Then,  $f(t) = t^2(u_0(t) - u_1(t)) + 1(u_1(t) - u_2(t)) + \cos(t-2)u_2(t).$

It is left as an exercise to compute the Laplace transform of  $f(t)$  from here. Note that, for instance, to find  $\mathcal{L}(t^2 u_1(t))$ , we want to use  $\mathcal{L}(u_a(t)f(t-a)) = e^{-sa}F(s)$  with  $a=1$  and  $f(t-1) = t^2$ ; then,  $f(t) = (t+1)^2 = t^2 + 2t + 1$  has Laplace transform  $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$ , and we combine to get  $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right).$   $\diamond$

**Example 180.** Solve the IVP  $x'' + 3x' + 2x = f(t)$ ,  $x(0) = x'(0) = 0$  with  $f(t) = \begin{cases} 1, & t \in [3, 4], \\ 0, & \text{otherwise.} \end{cases}$

**Solution.** First, we write  $f(t) = u_3(t) - u_4(t)$ . We can now take the Laplace transform of the DE to get

$$s^2 X(s) - sx(0) - x'(0) + 3(sX(s) - x(0)) + 2X(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s}) \frac{1}{s}.$$

Using that  $s^2 + 3s + 2 = (s+1)(s+2)$ , we find

$$X(s) = (e^{-3s} - e^{-4s}) \frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s}) \left[ \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \right],$$

where  $A, B, C$  are determined by partial fractions (and will be computed below). Taking the Laplace inverse of each of the six terms in this product, as in Example 178, we find

$$x(t) = A(u_3(t) - u_4(t)) + B(u_3(t)e^{-(t-3)} - u_4(t)e^{-(t-4)}) + C(u_3(t)e^{-2(t-3)} - u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as  $x(t) = \begin{cases} 0, & \text{if } t \leq 3, \\ A + B e^{-(t-3)} + C e^{-2(t-3)}, & \text{if } t \in [3, 4], \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}), & \text{if } t \geq 4. \end{cases}$

Finally,  $A = \frac{1}{(s+1)(s+2)} \Big|_{s=0} = \frac{1}{2}$ ,  $B = \frac{1}{s(s+2)} \Big|_{s=-1} = -1$ ,  $C = \frac{1}{s(s+1)} \Big|_{s=-2} = \frac{1}{2}$ . Check that these values make  $x(t)$  a continuous function (as it should be for physical reasons!).  $\diamond$

<sup>32</sup> The special case  $u_0(t)$  is also known as the Heaviside function, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that  $u_a(t) = u_0(t-a)$ .

## Boundary value problems and partial differential equations

### Endpoint problems and eigenvalues

**Example 181.** The IVP (initial value problem)  $y'' + 4y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 0$  has the unique solution  $y(x) = 0$ .  $\diamond$

Initial value problems are often used when the problem depends on time. Then, as we know for instance for the motion of a pendulum,  $y(0)$  and  $y'(0)$  describe the initial configuration at  $t = 0$ . For problems which instead depend on spatial variables, such as position, it may be natural to specify values at positions on the boundary (for instance, if  $y(x)$  describes the steady-state temperature of a rod at position  $x$ , we might know the temperature at the two end points).

The next two examples illustrate that such boundary value problem may or may not have unique solutions.

**Example 182.** The BVP (boundary value problem)  $y'' + 4y = 0$ ,  $y(0) = 0$ ,  $y(1) = 0$  has the unique solution  $y(x) = 0$ .

We know that the general solution to the DE is  $y(x) = A \cos(2x) + B \sin(2x)$ . The boundary conditions imply  $y(0) = A \stackrel{!}{=} 0$  and, already using that  $A = 0$ ,  $y(1) = B \sin(2) \stackrel{!}{=} 0$  shows that  $B = 0$  as well.  $\diamond$

**Example 183.** The BVP  $y'' + \pi^2 y = 0$ ,  $y(0) = 0$ ,  $y(1) = 0$  is solved by  $y(x) = B \sin(\pi x)$  for any value  $B$ .

This time, the general solution to the DE is  $y(x) = A \cos(\pi x) + B \sin(\pi x)$ . The boundary conditions imply  $y(0) = A \stackrel{!}{=} 0$  and, using that  $A = 0$ ,  $y(1) = B \sin(\pi) \stackrel{!}{=} 0$ . This second condition true for any  $B$ .  $\diamond$

It is therefore natural to ask: for which  $\lambda$  does  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$  have nonzero solutions? Such solutions are called **eigenfunctions** and  $\lambda$  is the corresponding **eigenvalue**.

**Remark 184.** Compare that to our previous use of the term eigenvalue: given a matrix  $A$ , we asked: for which  $\lambda$  does  $A\mathbf{v} - \lambda\mathbf{v} = 0$  have nonzero solutions  $\mathbf{v}$ ? Such solutions were called **eigenvectors** and  $\lambda$  was the corresponding **eigenvalue**.  $\diamond$

**Example 185.** Find all eigenfunctions and eigenvalues of  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$ .

Such a problem is called an **eigenvalue problem**.

**Solution.** The solutions of the DE look different in the cases  $\lambda < 0$ ,  $\lambda = 0$ ,  $\lambda > 0$ , so we consider them individually.

$\lambda = 0$ . Then  $y(x) = Ax + B$  and  $y(0) = y(L) = 0$  implies that  $y(x) = 0$ . No eigenfunction here.

$\lambda < 0$ . Write  $\lambda = -\rho^2$ . Then  $y(x) = Ae^{\rho x} + Be^{-\rho x}$ .  $y(0) = A + B \stackrel{!}{=} 0$  implies  $B = -A$ . Using that, we get  $y(L) = A(e^{\rho L} - e^{-\rho L}) \stackrel{!}{=} 0$ . For eigenfunctions we need  $A \neq 0$ , so  $e^{\rho L} = e^{-\rho L}$  which implies  $\rho L = -\rho L$ . This cannot happen since  $\rho \neq 0$  and  $L \neq 0$ . Again, no eigenfunctions in this case.

$\lambda > 0$ . Write  $\lambda = \rho^2$ . Then  $y(x) = A \cos(\rho x) + B \sin(\rho x)$ .  $y(0) = A \stackrel{!}{=} 0$ . Using that,  $y(L) = B \sin(\rho L) \stackrel{!}{=} 0$ . Since  $B \neq 0$  for eigenfunctions, we need  $\sin(\rho L) = 0$ . This happens if  $\rho L = n\pi$  for  $n = 0, 1, 2, \dots$ . Consequently, we do find the eigenfunctions  $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ ,  $n = 1, 2, 3, \dots$ , with eigenvalue  $\lambda = \left(\frac{n\pi}{L}\right)^2$ .  $\diamond$

**Review.** Eigenvalues and eigenfunctions of  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$ .  $\diamond$

**Example 186.** Suppose that a rod of length  $L$  is compressed by a force  $P$ . We model the shape of the rod by a function  $y(x)$  on the interval  $[0, L]$ . The theory of elasticity predicts that, under some simplifying assumptions,  $y$  should satisfy  $EIy'' + Py = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$ .

Here,  $EI$  is a constant modeling the inflexibility of the rod ( $E$ , known as Young's modulus, depends on the material, and  $I$  depends on the shape of cross-sections (it is the area moment of inertia)).

In other words,  $y'' + \lambda y = 0$ ,  $y(0) = 0$ ,  $y(L) = 0$ , with  $\lambda = \frac{P}{EI}$ . The fact that there is no nonzero solution unless  $\lambda = (\frac{\pi n}{L})^2$  for some  $n = 1, 2, 3, \dots$ , means that buckling can only occur if  $P = (\frac{\pi n}{L})^2 EI$ . In particular, no buckling occurs for forces less than  $\frac{\pi^2 EI}{L^2}$ . This is known as the critical load (or Euler load) of the rod.  $\diamond$

## The heat equation

We wish to describe one-dimensional heat flow<sup>33</sup>. Let  $u(x, t)$  describe the temperature at time  $t$  at position  $x$ . If we model a heated rod of length  $L$ , then  $x \in [0, L]$ .

Note that  $u(x, t)$  depends on two variables. When taking derivatives, we will use the notations  $u_t = \frac{\partial}{\partial t}u$  and  $u_{xx} = \frac{\partial^2}{\partial x^2}u$  for first and higher derivatives.

Experience tells us that heat flows from warmer to cooler areas and has an averaging effect. Make a sketch of some temperature profile  $u(x, t)$  for fixed  $t$ . Then as  $t$  increases, we expect maxima (where  $u_{xx} < 0$ ) of that profile to flatten out (so  $u_t < 0$ ); similarly, minima (where  $u_{xx} > 0$ ) should go up (so  $u_t > 0$ ). The simplest relationship between  $u_t$  and  $u_{xx}$  which conforms with our expectation is  $u_t = ku_{xx}$ , with  $k > 0$ . That's the [heat equation](#).

Note that the heat equation is a linear and homogeneous [partial differential equation](#).

In particular, the principle of superposition holds: if  $u_1$  and  $u_2$  solve the heat equation, then so does  $c_1u_1 + c_2u_2$ .

**Remark 187.** In higher dimensions, the heat equation takes the form  $u_t = k(u_{xx} + u_{yy})$  or  $u_t = k(u_{xx} + u_{yy} + u_{zz})$ . Note that  $\Delta u = u_{xx} + u_{yy} + u_{zz}$  is the Laplace operator<sup>34</sup>.  $\diamond$

**Example 188.** Note that  $u(x, t) = ax + b$  solves the heat equation.  $\diamond$

Let us think about what is needed to describe a unique solution of the heat equation.

Initial condition at  $t = 0$ :  $u(x, 0) = f(x)$  [temperature distribution at time  $t = 0$ ]  
 Boundary condition at  $x = 0$  and  $x = L$ : [heat only enters/exits at boundary]  
 for instance,  $u(0, t) = u(L, t) = 0$  [by adding  $ax + b$  to  $u$  this covers any constant boundary values]  
 [another option are boundary conditions like  $u_x(0, t) = u_x(L, t) = 0$ , which would model insulated ends]

**Example 189.** To get a feeling, let us find some other solutions to  $u_t = u_{xx}$  (for starters,  $k = 1$ ).

For instance,  $u(x, t) = e^t e^x$  is a solution. Not a very interesting one for modeling heat flow because it increases exponentially in time.

More interesting are  $u(x, t) = e^{-t} \cos(x)$  and  $u(x, t) = e^{-t} \sin(x)$ . More generally,  $e^{-n^2 t} \cos(nx)$  and  $e^{-n^2 t} \sin(nx)$  are solutions. This actually reveals a strategy for solving the heat equation with conditions such as  $u_t = u_{xx}$ ,  $u(0, t) = 0$ ,  $u(L, t) = 0$ ,  $u(x, 0) = f(x)$ .

Namely, the solutions  $u_n(x, t) = e^{-n^2 t} \sin(nx)$  all satisfy  $u(0, t) = 0$ ,  $u(\pi, t) = 0$ . On the other hand,  $u_n(x, 0) = \sin(nx)$ . To find  $u(x, t)$  such that  $u(x, 0) = f(x)$ , we thus only need to write  $f(x)$  as a Fourier (sine) series.  $\diamond$

33. If this sounds very specialized, it might help to know that the heat equation is also used, for instance, in probability (Brownian motion), financial math (Black-Scholes), or chemical processes (diffusion equation).

34. The Laplacian  $\Delta u$  is also often written as  $\Delta u = \nabla^2 u$ . The operator  $\nabla = (\partial/\partial x, \partial/\partial y)$  is pronounced "nabla" (Greek for a certain harp) or "del" (Persian for heart), and  $\nabla^2$  is short for the inner product  $\nabla \cdot \nabla$ .

**Review.** heat equation ◇

**Example 190.** Find the unique solution to:

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u(0, t) &= u(L, t) = 0 && \text{(BC)} \\ u(x, 0) &= f(x), \quad x \in (0, L) && \text{(IC)} \end{aligned}$$

**Solution.** We will first look for simple solutions of PDE+BC (and then we plan to take a superposition of such solutions that satisfies IC as well). Namely, we look for solutions  $u(x, t) = X(x)T(t)$ . This approach is called [separation of variables](#) and it is crucial for solving other PDEs as well.

Plugging into the PDE, we get  $X(x)T'(t) = kX''(x)T(t)$ , and so  $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$ . Note that the two sides cannot depend on  $x$  (because the right-hand side doesn't) and they cannot depend on  $t$  (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant  $-\lambda$ . Then,  $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$ .

We thus have  $X'' + \lambda X = 0$  and  $T' + \lambda k T = 0$ .

Consider the BC.  $u(0, t) = X(0)T(t) = 0$  implies  $X(0) = 0$  (because otherwise  $T(t) = 0$  for all  $t$ , which would mean that  $u(x, t)$  is the dull zero solution). Likewise,  $u(L, t) = X(L)T(t) = 0$  implies  $X(L) = 0$ .

So  $X$  solves  $X'' + \lambda X = 0$ ,  $X(0) = 0$ ,  $X(L) = 0$ . We know that, up to multiples, the only nonzero solutions are the eigenfunctions  $X(x) = \sin\left(\frac{\pi n}{L}x\right)$  corresponding to the eigenvalues  $\lambda = \left(\frac{\pi n}{L}\right)^2$ ,  $n = 1, 2, 3, \dots$

On the other hand,  $T$  solves  $T' + \lambda k T = 0$ , and hence  $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$ .

Taken together, we have the solutions  $u_n(x, t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right)$  solving PDE+BC.

We wish to combine these in such a way that IC holds. At  $t = 0$ ,  $u_n(x, 0) = \sin\left(\frac{\pi n}{L}x\right)$ . All of these are  $2L$ -periodic. Hence, we extend  $f(x)$ , which is only given on  $(0, L)$ , to an odd  $2L$ -periodic function. By making it odd, its Fourier series will only involve sine terms:  $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L}x\right)$ .

Consequently, PDE+BC+IC is solved by  $u(x, t) = \sum_{n=1}^{\infty} b_n u_n(x, t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L}x\right)$ . ◇

**Example 191.** Find the unique solution to:

$$\begin{aligned} u_t &= u_{xx} \\ u(0, t) &= u(1, t) = 0 \\ u(x, 0) &= 1, \quad x \in (0, 1) \end{aligned}$$

**Solution.** This is the case  $k = 1$ ,  $L = 1$  and  $f(x) = 1$ ,  $x \in (0, 1)$ , of the previous example.

In the final step, we extend  $f(x)$  to the 2-periodic odd function of Example 152. In particular, we have already computed that the Fourier series is  $f(x) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{\pi n} \sin(\pi n x)$ .

Hence,  $u(x, t) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(\pi n x)$ . ◇

The boundary conditions in the next example model insulated ends.

**Example 192.** Find the unique solution to:

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u_x(0, t) &= u_x(L, t) = 0 && \text{(BC)} \\ u(x, 0) &= f(x), \quad x \in (0, L) && \text{(IC)} \end{aligned}$$

**Solution.** We proceed as before and look for solutions  $u(x, t) = X(x)T(t)$ . Plugging into the PDE, we get  $X(x)T'(t) = kX''(x)T(t)$ , and so  $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$ . We thus have  $X'' + \lambda X = 0$  and  $T' + \lambda k T = 0$ .

From the BC  $u_x(0, t) = X'(0)T(t) = 0$  we get  $X'(0) = 0$ . Likewise,  $u_x(L, t) = X'(L)T(t) = 0$  implies  $X'(L) = 0$ . So  $X$  solves  $X'' + \lambda X = 0$ ,  $X'(0) = 0$ ,  $X'(L) = 0$ . It is left as an exercise (DO IT!!) that, up to multiples, the only nonzero solutions of this eigenvalue problem are  $X(x) = \cos\left(\frac{\pi n}{L}x\right)$  corresponding to  $\lambda = \left(\frac{\pi n}{L}\right)^2$ ,  $n = 0, 1, 2, 3, \dots$

As before,  $T$  solves  $T' + \lambda k T = 0$ , and hence  $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$ .

Taken together, we have the solutions  $u_n(x, t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L}x\right)$  solving PDE+BC.

We wish to combine these in such a way that IC holds. At  $t = 0$ ,  $u_n(x, 0) = \cos\left(\frac{\pi n}{L}x\right)$ . All of these are  $2L$ -periodic. Hence, we extend  $f(x)$ , which is only given on  $(0, L)$ , to an even  $2L$ -periodic function. By making it even, its Fourier series will only involve cosine terms:  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n}{L}x\right)$ .

So, PDE+BC+IC is solved by  $u(x, t) = \frac{a_0}{2} u_0(x, t) + \sum_{n=1}^{\infty} a_n u_n(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \cos\left(\frac{\pi n}{L}x\right)$ . ◇

**Problem 1.** Without solving the equation  $(y - 2)y' = x + e^y + 1$ , answer the following questions.

- (a) Does the existence/uniqueness theorem guarantee the existence of a solution to the above equation with initial condition  $y(2) = 0$ ? If so, does it guarantee the solution to be unique?
- (b) Same question for the initial condition  $y(0) = 2$ .
- (c) Sketch the slope field of the differential equation. What does it suggest regarding the previous questions?
- (d) Consider the solution with initial condition  $y(1) = 0$ . Find the equation for its tangent line at the point  $(1, 0)$ .
- (e) Again, considering the solution with initial condition  $y(1) = 0$ , what is  $y''(1)$ ?

**Problem 2.** Solve the initial value problem  $y' = 2xy + 3x^2 e^{x^2}$ ,  $y(0) = 5$ .

**Problem 3.** Find a general solution of the equation  $x(x + y)y' = y(3x + y)$ .

**Problem 4.** Find a general solution of the equation  $2 + \frac{dy}{dx} = \sqrt{2x + y}$ .

**Problem 5.** In a city with a fixed population  $P$ , the time rate of change of the number  $N$  of people who have heard a certain rumor is proportional to  $N$  and  $P - N$ . Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

**Problem 6.** Solve the initial value problem  $y'' - 5y' + 6y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$ .

**Problem 7.** Solve:  $x^2 \frac{dy}{dx} = xy - x^2 e^{y/x}$ ,  $y(1) = 0$

**Problem 8.** Find a general solution of the equation  $xy' = y + x^2 \cos(x)$ .

**Problem 9.** Find the general solution to  $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$ .

**Problem 10.** Write down a homogeneous linear differential equation satisfied by  $y(x) = 1 - 5x^2 e^{-2x}$ .

**Problem 1.** Without solving the equation  $(y - 2)y' = x + e^y + 1$ , answer the following questions.

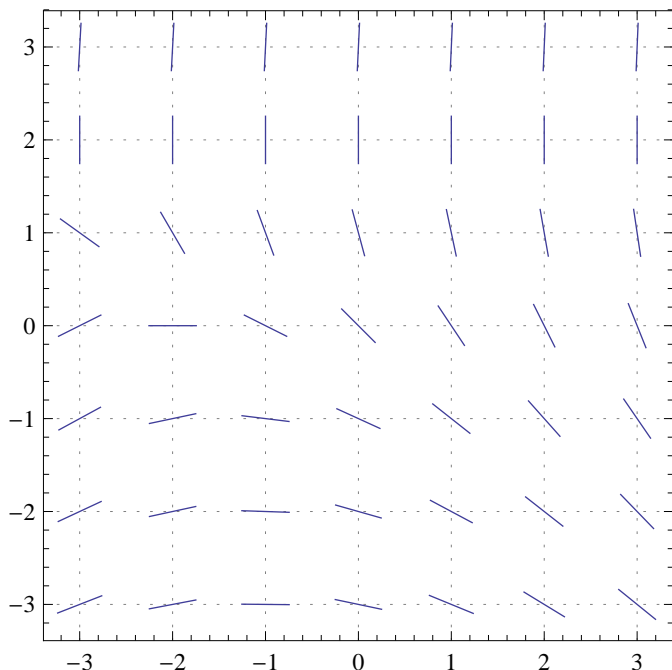
- Does the existence/uniqueness theorem guarantee the existence of a solution to the above equation with initial condition  $y(2) = 0$ ? If so, does it guarantee the solution to be unique?
- Same question for the initial condition  $y(0) = 2$ .
- Sketch the slope field of the differential equation. What does it suggest regarding the previous questions?
- Consider the solution with initial condition  $y(1) = 0$ . Find the equation for its tangent line at the point  $(1, 0)$ .
- Again, considering the solution with initial condition  $y(1) = 0$ , what is  $y''(1)$ ?

**Solution.** Here,  $y' = f(x, y)$  with  $f(x, y) = \frac{x + e^y + 1}{y - 2}$ .

- The function  $f(x, y)$  is continuous at all points  $(x, y)$  for which  $y \neq 2$ . This includes the point  $(2, 0)$  and a neighborhood of it. Hence there exists a solution with initial condition  $y(2) = 0$ .

$\frac{\partial}{\partial y} f(x, y) = \frac{e^y(y - 2) - (x + e^y + 1)}{(y - 2)^2}$  is again continuous at all points  $(x, y)$  for which  $y \neq 2$ . Hence the solution with initial condition  $y(2) = 0$  is unique.

- The function  $f(x, y)$  is not continuous at the point  $(0, 2)$ . Hence the existence/uniqueness theorem does not guarantee the existence of a solution with initial condition  $y(0) = 2$ .
- (If we look very carefully; sorry for that) the slope field below suggests that, if we permit the slope  $y'(0)$  to be infinite, there should be two solutions to the DE with initial condition  $y(0) = 2$  (very much like in the example  $y' = -x/y$ ,  $y(a) = 0$ , from class where we obtained half-circles as solution functions). (If you are a strict person, it would also be OK to say that no solution exists because  $y'(0)$  would be undefined, whereas a solution to a differential equation should be differentiable. The important part is to be aware of the issues.)



- (d) Plugging  $y(1)=0$  into the equation yields  $-2y'(1)=1+1+1$ . Hence  $y'(1)=-\frac{3}{2}$  and the tangent line at  $(1,0)$  has equation  $y=-\frac{3}{2}(x-1)$ .
- (e) Apply  $\frac{d}{dx}$  to the DE to get  $y'y'+(y-2)y''=1+e^y y'$ . Using  $y(1)=0$  and  $y'(1)=-\frac{3}{2}$ , we get  $(-\frac{3}{2})^2-2y''(1)=1-\frac{3}{2}$  and hence  $y''(1)=\frac{11}{8}$ .  $\square$

**Problem 2.** Solve the initial value problem  $y'=2xy+3x^2e^{x^2}$ ,  $y(0)=5$ .

**Solution.** This equation is linear:  $y'-2xy=3x^2e^{x^2}$

The integrating factor is  $e^{\int -2x dx} = e^{-x^2}$ :

$$\begin{aligned} e^{-x^2}y' - e^{-x^2}2xy &= 3x^2 \\ \frac{d}{dx}[e^{-x^2}y] &= \frac{d}{dx}[x^3] \\ e^{-x^2}y &= x^3 + C \end{aligned}$$

Using  $y(0)=5$ , we find  $5=C$ . Hence the solution is  $y=(x^3+5)e^{x^2}$ .  $\square$

**Problem 3.** Find a general solution of the equation  $x(x+y)y'=y(3x+y)$ .

**Solution.** This equation is homogeneous as can be seen by dividing by  $xy$  to get  $\left(\frac{x}{y}+1\right)y'=3+\frac{y}{x}$ .

Substituting  $u=\frac{y}{x}$  gives  $\frac{dy}{dx}=u+x\frac{du}{dx}$  and hence:

$$\begin{aligned} (u^{-1}+1)(u+xu') &= 3+u \\ (1+u^{-1})u' &= \frac{2}{x} \\ u + \ln|u| &= 2\ln|x| + C \\ |u|e^u &= e^C x^2 \end{aligned}$$

Assuming  $u > 0$ , we get  $ue^u = Dx^2$  with  $D > 0$ . This implicit solution cannot be made more explicit using standard functions. For the original problem, we get the implicit solution  $ye^{y/x} = Dx^3$ .  $\square$

**Problem 4.** Find a general solution of the equation  $2 + \frac{dy}{dx} = \sqrt{2x+y}$ .

**Solution.** Substitute  $u=2x+y$ . Then  $\frac{dy}{dx} = \frac{du}{dx} - 2$  and we get  $\frac{du}{dx} = \sqrt{u}$  which is separable.

$2u^{1/2} = x + C$ , hence  $u = \frac{1}{4}(x+C)^2$  and  $y = u - 2x = \frac{1}{4}(x+C)^2 - 2x$  [which is a solution as long as  $x+C > 0$ ].  $\square$

**Problem 5.** In a city with a fixed population  $P$ , the time rate of change of the number  $N$  of people who have heard a certain rumor is proportional to  $N$  and  $P-N$ . Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

**Solution.**  $\frac{dN}{dt} = kN(P-N)$ .  $N(0)=0.1P$  and  $N(1)=0.2P$ . We need  $N(2)$ .

$$\int \frac{dN}{N(P-N)} = \frac{1}{P} \int \left[ \frac{1}{N} + \frac{1}{P-N} \right] dN = \int k dt$$

Hence  $\frac{1}{P} \ln \left| \frac{N}{P-N} \right| = kt + C$ . Using  $N(0)=0.1P$  and  $N(1)=0.2P$ , we have  $\frac{1}{P} \ln \frac{1}{9} = C$  and  $\frac{1}{P} \ln \frac{1}{4} = k + C$ .

This gives  $\ln \left| \frac{N}{P-N} \right| = \left( \ln \frac{9}{4} \right) t + \ln \frac{1}{9}$  and hence  $\frac{N}{P-N} = \frac{1}{9} \left( \frac{9}{4} \right)^t$ .

When  $t=2$  thus  $\frac{N(2)}{P-N(2)} = \frac{9}{16}$  and, solving for  $N(2)$ , we find  $\frac{25}{16}N(2) = \frac{9}{16}P$ , thus  $N(2) = \frac{9}{25}P$  which is 36%.  $\square$

**Problem 6.** Solve the initial value problem  $y'' - 5y' + 6y = 0$ ,  $y(0) = 0$ ,  $y'(0) = 1$ .

**Solution.** This is a linear homogeneous equation with constant coefficients. The characteristic equation is  $r^2 - 5r + 6 = (r-2)(r-3)$ . Hence the general solution is  $y(x) = c_1 e^{2x} + c_2 e^{3x}$ . Solving

$$\begin{aligned} y(0) = 0 &= c_1 + c_2 \\ y'(0) = 1 &= 2c_1 + 3c_2 \end{aligned}$$

gives  $c_1 = -1$  and  $c_2 = 1$ . The solution is  $y(x) = e^{3x} - e^{2x}$ .  $\square$

**Problem 7.** Solve:  $x^2 \frac{dy}{dx} = xy - x^2 e^{y/x}$ ,  $y(1) = 0$

**Solution.**  $\frac{dy}{dx} = \frac{y}{x} - e^{y/x}$  is homogeneous. Substitute  $u = \frac{y}{x}$  to get  $u + x \frac{du}{dx} = u - e^u$  or  $x \frac{du}{dx} = -e^u$ .

By separation of variables  $e^{-u} = \ln|x| + C$ . Using  $u(1) = \frac{0}{1} = 0$  we find  $C = 1$ .

Hence  $-u = \ln(1 + \ln x)$  and therefore  $y = ux = -x \ln(1 + \ln x)$ .  $\square$

**Problem 8.** Find a general solution of the equation  $xy' = y + x^2 \cos(x)$ .

**Solution.** This equation is linear:  $y' - \frac{1}{x}y = x \cos(x)$

The integrating factor is  $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$ :

$$\begin{aligned} \frac{1}{x} y' - \frac{1}{x^2} y &= \cos(x) \\ \frac{d}{dx} \left[ \frac{1}{x} y \right] &= \frac{d}{dx} [\sin(x)] \\ \frac{y}{x} &= \sin(x) + C \end{aligned}$$

Hence a general solution is  $y = x \sin(x) + Cx$ .  $\square$

**Problem 9.** Find the general solution to  $y^{(5)} - 4y^{(4)} + 5y''' - 2y'' = 0$ .

**Solution.** This is a linear homogeneous equation with constant coefficients. The characteristic equation is  $r^5 - 4r^4 + 5r^3 - 2r^2 = r^2(r-1)^2(r-2)$ . Hence the general solution is

$$y(x) = c_1 + c_2 x + (c_3 + c_4 x)e^x + c_5 e^{2x}.$$

$\square$

**Problem 10.** Write down a homogeneous linear differential equation satisfied by  $y(x) = 1 - 5x^2 e^{-2x}$ .

**Solution.** A linear homogeneous DE with constant coefficients will do if its characteristic equation is  $r(r+2)^3 = 0$ . Since  $r(r+2)^3 = r^4 + 6r^3 + 12r^2 + 8r$ , a corresponding DE is

$$y^{(4)} + 6y''' + 12y'' + 8y' = 0.$$

$\square$



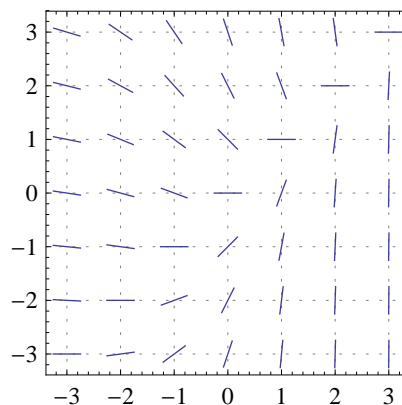
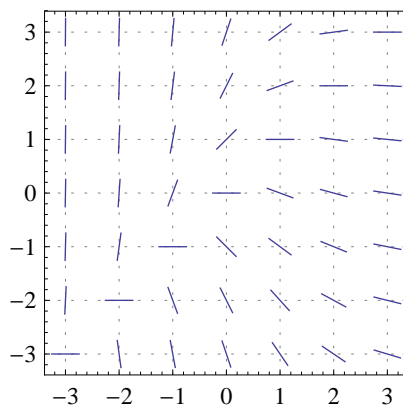
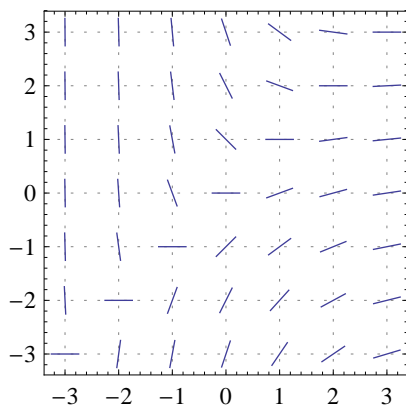
# Midterm #1

MATH 286 — Differential Equations Plus  
Thursday, February 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

**Problem 1. (5 points)** Circle the slope field below which belongs to the differential equation  $e^x y' = x - y$ .



**Problem 2. (20 points)** Find the general solution to the differential equation  $y^{(5)} - 4y^{(4)} + 4y^{(3)} = 0$ .

$y(x) =$

**Problem 3. (20 points)** Solve the initial value problem

$$(x^2 + 1) \frac{dy}{dx} + xy = \frac{1}{\sqrt{x^2 + 1}}, \quad y(0) = 1.$$

$y(x) =$

**Problem 4. (20 points)** The time rate of change of a rabbit population  $P$  is proportional to the square root of  $P$ . At time  $t = 0$ , the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be after two months?

$P(2 \text{ months}) =$

**Problem 5. (20 points)** For each  $c \geq 0$ , let  $y_c(x) = \begin{cases} x^3, & \text{if } x < 0, \\ 0, & \text{if } 0 \leq x \leq c, \\ (x - c)^3, & \text{if } x > c. \end{cases}$

- (a) Sketch the graph of  $y_c(x)$  for  $c = 3/2$ .
- (b) Show that, for all  $c \geq 0$ ,  $y_c$  is a solution to the initial value problem

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(0) = 0.$$

- (c) Explain why (b) does not contradict the theorem on existence and uniqueness for solutions to initial value problems.

**Problem 6. (20 points)** Find a general solution to the differential equation

$$x^2 \frac{dy}{dx} - x^2 - y^2 - 3xy = 0.$$

$y(x) =$

**Problem 7. (5 points)** Consider the differential equation

*Hint:* Do not attempt to solve the DE.

$$y' = y^4 + x^4 + 1.$$

Is it possible that there exists a solution with the property that  $\lim_{x \rightarrow \infty} y(x) = -\infty$ ? Why, or why not?

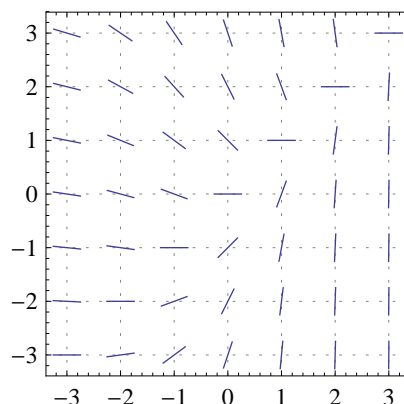
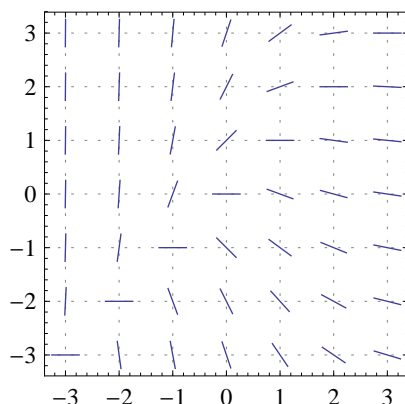
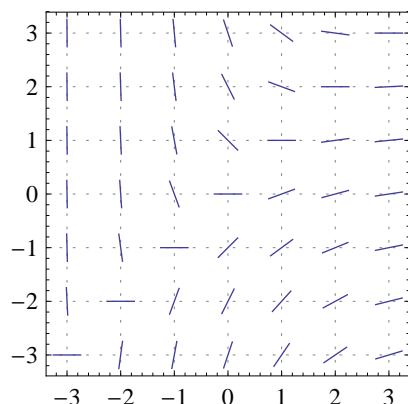
# Midterm #1

MATH 286 — Differential Equations Plus  
Thursday, February 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

**Problem 1. (5 points)** Circle the slope field below which belongs to the differential equation  $e^x y' = x - y$ .



**Problem 2. (20 points)** Find the general solution to the differential equation  $y^{(5)} - 4y^{(4)} + 4y^{(3)} = 0$ .

**Solution.** This is a linear homogeneous equation with constant coefficients. The characteristic polynomial is  $r^5 - 4r^4 + 4r^3 = r^3(r - 2)^2$ .

Hence, the general solution is  $c_1 + c_2x + c_3x^2 + (c_4 + c_5x)e^{2x}$ . □

**Problem 3. (20 points)** Solve the initial value problem

$$(x^2 + 1) \frac{dy}{dx} + xy = \frac{1}{\sqrt{x^2 + 1}}, \quad y(0) = 1.$$

**Solution.** Dividing by  $x^2 + 1$ , we get

$$\frac{dy}{dx} + \frac{x}{x^2 + 1} y = \frac{1}{(x^2 + 1)^{3/2}},$$

which is a linear equation with integrating factor

$$e^{\int \frac{x}{x^2 + 1} dx} = e^{\frac{1}{2} \ln(x^2 + 1)} = (x^2 + 1)^{1/2}.$$

Multiply through by this integrating factor, we obtain

$$(x^2 + 1)^{1/2} \frac{dy}{dx} + \frac{x}{(x^2 + 1)^{1/2}} y = \frac{d}{dx} ((x^2 + 1)^{1/2} y) = \frac{1}{x^2 + 1}.$$

Integrating both sides with respect to  $x$  yields

$$(x^2 + 1)^{1/2} y(x) = \arctan(x) + C.$$

Since  $y(0) = 1$ , we find  $C = 1$ . Therefore,

$$y(x) = \frac{\arctan(x) + 1}{(x^2 + 1)^{1/2}}.$$

□

**Problem 4. (20 points)** The time rate of change of a rabbit population  $P$  is proportional to the square root of  $P$ . At time  $t = 0$ , the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be after two months?

**Solution.**  $\frac{dP}{dt} = k\sqrt{P}$  and  $P(0) = 100$ ,  $P'(0) = 20$ . The problem asks for  $P(2)$ .

At  $t = 0$ , we have  $20 = P'(0) = k\sqrt{P(0)} = 10k$ . Hence  $k = 2$ .

By separation of variables, we find  $2\sqrt{P} = 2t + C$ . From  $P(0) = 100$ ,  $C = 20$ . Thus,  $P = (t + 10)^2$ .

We conclude that there are  $P(2) = 12^2 = 144$  rabbits after two months.

□

**Problem 5. (20 points)** For each  $c \geq 0$ , let  $y_c(x) = \begin{cases} x^3, & \text{if } x < 0, \\ 0, & \text{if } 0 \leq x \leq c, \\ (x - c)^3, & \text{if } x > c. \end{cases}$

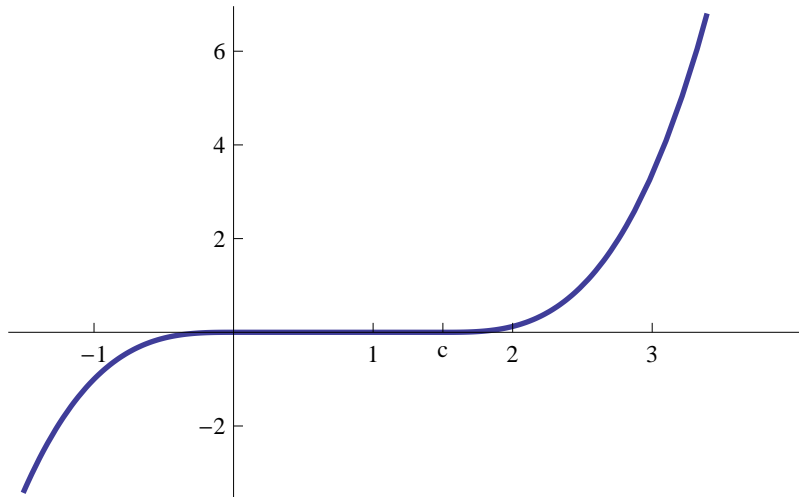
- (a) Sketch the graph of  $y_c(x)$  for  $c = 3/2$ .
- (b) Show that, for all  $c \geq 0$ ,  $y_c$  is a solution to the initial value problem

$$\frac{dy}{dx} = 3y^{2/3}, \quad y(0) = 0.$$

- (c) Explain why (b) does not contradict the theorem on existence and uniqueness for solutions to initial value problems.

**Solution.**

- (a) For  $c = 3/2$ , the graph looks as follows.



- (b) Clearly,  $y_c(0) = 0$ . Moreover, since  $\frac{d}{dx}x^3 = 3x^2 = 3(x^3)^{2/3}$ ,  $\frac{d}{dx}(x - c)^3 = 3(x - c)^2 = 3[(x - c)^3]^{2/3}$ , and all the left and right derivatives at the appropriate endpoints match,  $y_c(x)$  satisfies the DE  $y' = 3y^{2/3}$ .
- (c) Note that (b) demonstrates that there are infinitely many solutions to the IVP. Thus we do not have uniqueness.

Write the DE as  $y' = f(x, y)$  with  $f(x, y) = 3y^{2/3}$  which is continuous around the point  $(0, 0) = (0, y(0))$ . However,  $\frac{\partial}{\partial y}f(x, y) = 2y^{-1/3}$  is not continuous around the point  $(0, 0)$ . Our theorem on existence and uniqueness therefore only guarantees existence but not uniqueness for this particular IVP.  $\square$

**Problem 6. (20 points)** Find a general solution to the differential equation

$$x^2 \frac{dy}{dx} - x^2 - y^2 - 3xy = 0.$$

**Solution.** Dividing through by  $x^2$ , the equation becomes

$$\frac{dy}{dx} = 1 + (y/x)^2 + 3(y/x).$$

Making the homogeneous substitution  $v = y/x$ , we have  $\frac{dy}{dx} = x \frac{dv}{dx} + v = 1 + v^2 + 3v$ , which gives

$$x \frac{dv}{dx} = v^2 + 2v + 1 = (v + 1)^2.$$

By separation of variables, we get  $\frac{dv}{(v+1)^2} = \frac{dx}{x}$  (note that we just lost the singular solution  $v = -1$  which corresponds to the solution  $y = -x$  of the original equation). Integrating, we have  $-(v+1)^{-1} = \ln|x| + C$  and hence

$$y(x) = -\frac{x}{\ln|x| + C} - x.$$

□

**Problem 7. (5 points)** Consider the differential equation

*Hint:* Do not attempt to solve the DE.

$$y' = y^4 + x^4 + 1.$$

Is it possible that there exists a solution with the property that  $\lim_{x \rightarrow \infty} y(x) = -\infty$ ? Why, or why not?

**Solution.** It is impossible. It follows from the differential equation that  $y'(x) > 0$  for all  $x$ . In other words, any solution  $y(x)$  is an increasing function, and thus cannot approach  $-\infty$  for large  $x$ . □



Student Name: \_\_\_\_\_  
Student Net ID: \_\_\_\_\_

MATH 286 SECTION X1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 1

September 25, 2013

INSTRUCTOR: M. BRANNAN

**INSTRUCTIONS**

- This exam 60 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **EXPLAIN YOUR WORK!** Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- **GOOD LUCK!**

**PLEASE NOTE:** “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

Question:	1	2	3	4	5	Total
Points:	6	9	10	8	7	40
Score:						

1. (6 points) Solve the initial value problem

$$\frac{dy}{dx} = y^{-1}e^y \cos x, \quad y(0) = 1.$$

An implicit expression for the solution is fine.

**Solution:** This is a separable equation:

$$ye^{-y}dy = \cos x dx \implies \int ye^{-y}dy = \int \cos x dx + C \implies -ye^{-y} - e^{-y} = \sin x + C.$$

To find  $C$ , we plug in  $y(0) = 1$ , which gives  $-2e^{-1} = C$ . The solution is therefore

$$ye^{-y} + e^{-y} = 2e^{-1} - \sin x.$$

2. Consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x} + x \ln x \quad (x > 0).$$

- (a) (3 points) Without explicitly solving this ODE, determine whether the corresponding initial value problem with initial condition  $y(1) = 0$  has a unique solution, no solution, or more than one solution. *Explain your answer!*

**Solution:** Let  $F(x, y) = \frac{y}{x} + x \ln x$ . According to the existence-uniqueness theorem for initial value problems, there exists a unique solution to the above IVP if  $F$  and  $\frac{\partial F}{\partial y} = \frac{1}{x}$  are both continuous on a rectangle containing the initial data  $(1, y(1)) = (1, 0)$ . Since this is indeed the case, there is a unique solution.

- (b) (6 points) Find the solution to the IVP

$$\frac{dy}{dx} = \frac{y}{x} + x \ln x, \quad y(1) = 1.$$

**Solution:** We can write this equation as  $\frac{dy}{dx} - \frac{y}{x} = x \ln x$ , which is obviously linear, with integrating factor

$$e^{P(x)} = e^{\int -x^{-1} dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying through by this integrating factor, we get

$$\begin{aligned} \frac{1}{x} \frac{dy}{dx} - \frac{1}{x^2} y &= \ln x \iff \frac{d}{dx} \left( \frac{y}{x} \right) = \ln x \\ \implies \frac{y}{x} &= \int \ln x dx + C = x \ln x - x + C \\ \implies y &= x^2 \ln x - x^2 + Cx. \end{aligned}$$

Plugging in the initial conditions we get  $C = 2$ .

3. Consider the ODE

$$\frac{x}{x^2 + y^2 + 1} - \sin x + \left( \frac{y}{x^2 + y^2 + 1} \right) \frac{dy}{dx} = 0 \quad ((x, y) \neq (0, 0)).$$

(a) (4 points) Show that this ODE is exact.

**Solution:** Let  $M(x, y) = \frac{x}{x^2 + y^2 + 1} - \sin x$  and  $N(x, y) = \frac{y}{x^2 + y^2 + 1}$ . Then

$$M_y = \frac{-2xy}{(x^2 + y^2 + 1)^2} \quad \& \quad N_x = \frac{-2xy}{(x^2 + y^2 + 1)^2}.$$

Since  $N_x = M_y$ , the equation is exact.

(b) (6 points) Find an implicit expression for the general solution to this ODE.

**Solution:** We need to find a function  $F(x, y)$ , defined for  $(x, y) \neq (0, 0)$ , such that our solution curves lie along the level sets  $F(x, y) = C$ . If this is the case, then we must have  $M = F_x$  and  $N = F_y$ . Solving these equations, we obtain

$$\begin{aligned} F(x, y) &= \int M(x, y) dx = \int \left( \frac{x}{x^2 + y^2 + 1} - \sin x \right) dx \\ &= \frac{1}{2} \ln(x^2 + y^2 + 1) + \cos x + g(y), \end{aligned}$$

where  $g(y)$  is some unknown function of  $y$ . To find  $g(y)$ , we use the equation  $F_y = N$ , which gives

$$N = \frac{y}{x^2 + y^2 + 1} = F_y = \frac{y}{x^2 + y^2 + 1} + g'(y) \implies g'(y) = 0.$$

Therefore  $g(y) = K$  is constant and our solutions are on the level curves

$$F(x, y) = \frac{1}{2} \ln(x^2 + y^2 + 1) + \cos x + K = C.$$

4. (8 points) Find the general solution to the second order ODE

$$xy'' = x \exp\left(\frac{y'}{x}\right) + y' \quad (x > 0),$$

where  $\exp a = e^a$ . (**Note:** It is OK if your final answer involves an indefinite integral that cannot be evaluated).

**Solution:** Making the substitution  $z = y'$ , the equation becomes first order:

$$xz' = x \exp\left(\frac{z}{x}\right) + z \iff z' = \exp\left(\frac{z}{x}\right) + \frac{z}{x}.$$

This is now a homogeneous substitution problem. Set  $v = \frac{z}{x}$ . Then  $z' = xv' + v$ , which gives

$$xv' + v = e^v + v \implies e^{-v}dv = \frac{dx}{x} \implies -e^{-v} = \ln x + C.$$

Solving for  $v$ , we get

$$\frac{z}{x} = v = -\ln(-\ln x - C) \implies z = -x \ln(-\ln x - C)$$

To get  $y$ , we integrate:

$$y(x) = \int z(x)dx + K = \int -x \ln(-\ln x + C)dx + K.$$

5. (7 points) The functions

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = e^{2x} \quad (x \in \mathbb{R}),$$

are solutions to the 3rd order linear ODE

$$y''' + Ay'' + By' + Cy = 0.$$

What are  $A, B, C$ ?

**Solution:** The characteristic polynomial for this ODE is  $P(r) = r^3 + Ar^2 + Br + C$ . On the other hand, since  $y_1, y_2, y_3$  are solutions, we know that  $P(r)$  has roots 1,  $-1$ , and 2 (each with multiplicity 1). Thus

$$P(r) = (r - 1)(r + 1)(r - 2) = (r^2 - 1)(r - 2) = r^3 - 2r^2 - r + 2.$$

Comparing coefficients, we obtain

$$A = -2, \quad B = -1, \quad C = 2.$$

*(Extra work space.)*

**Problem 1.** Solve  $y'' + 2y' + y = 2e^{2x} + e^{-x}$ ,  $y(0) = -1$ ,  $y'(0) = 2$ .

**Problem 2.**

- (a) Assume that the angle  $\theta(t)$  of a swinging pendulum is described by  $\theta'' + 4\theta = 0$ . Suppose  $\theta(0) = \frac{3}{10}$  and  $\theta'(0) = -\frac{4}{5}$ . What is the amplitude of the resulting periodic oscillations?
- (b) For which values of the damping constant  $c > 0$  is the system  $y'' + cy' + 5y = 0$  underdamped?
- (c) For which value of the external frequency  $\omega$  does the system  $y'' + 4y = 3\cos(\omega x)$  exhibit resonance?
- (d) A forced mechanical oscillator is described by  $x'' + 2x' + x = 25\cos(2t)$ . What is the amplitude of the resulting steady periodic oscillations?

**Problem 3.** The motion of a certain mass on a spring is described by  $x'' + x' + \frac{x}{2} = 5\sin(\omega t)$ .

- (a) Assume first that  $\omega = 1$ . Find the position function  $x(t)$  if  $x(0) = 2$  and  $x'(0) = 0$ .
- (b) Suppose the frequency of the external force is changed so that  $x'' + x' + \frac{x}{2} = 5\sin(\omega t)$ . Does practical resonance occur for some value of  $\omega$ ? If so, for what  $\omega$ ?

**Problem 4.** Find the general solution of  $y'' - 4y' + 4y = 3e^{2x}$ .

**Problem 5.**

- (a) Consider the differential equation  $x^2y'' - 4xy' + 6y = 0$ . Find all solutions of the form  $y(x) = x^r$ .
- (b) Show that the solutions you found are independent.
- (c) Note that the Wronskian of your solutions is zero for  $x = 0$ . Why does this not contradict the independence?
- (d) Find the general solution of  $x^2y'' - 4xy' + 6y = x^3$ .

**Problem 6.** Solve  $\mathbf{x}' = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} \mathbf{x}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$ .



# Preparing Midterm #2

MATH 286 — Differential Equations Plus

March 13, 2014

**Problem 1.** Solve  $y'' + 2y' + y = 2e^{2x} + e^{-x}$ ,  $y(0) = -1$ ,  $y'(0) = 2$ .

**Solution.** The characteristic equation for the associated homogeneous DE has roots  $-1, -1$  (the “old” roots). The right-hand side solves a DE whose characteristic equation has roots  $-1, 2$  (the “new” roots).

Hence, there is a particular solution of the form  $y_p = Ae^{2x} + Bx^2e^{-x}$ . To find  $A, B$  we plug into the DE. [...] We find  $A = \frac{2}{9}$  and  $B = \frac{1}{2}$ .

Particular solution:  $y_p = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x}$

General solution:  $y = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}$

Now, we use the initial values [...], to find  $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$ . □

**Problem 2.**

- (a) Assume that the angle  $\theta(t)$  of a swinging pendulum is described by  $\theta'' + 4\theta = 0$ . Suppose  $\theta(0) = \frac{3}{10}$  and  $\theta'(0) = -\frac{4}{5}$ . What is the amplitude of the resulting periodic oscillations?
- (b) For which values of the damping constant  $c > 0$  is the system  $y'' + cy' + 5y = 0$  underdamped?
- (c) For which value of the external frequency  $\omega$  does the system  $y'' + 4y = 3\cos(\omega x)$  exhibit resonance?
- (d) A forced mechanical oscillator is described by  $x'' + 2x' + x = 25\cos(2t)$ . What is the amplitude of the resulting steady periodic oscillations?

**Solution.**

- (a) The characteristic equation has roots  $\pm 2i$ . Hence,  $\theta(t) = A\cos(2t) + B\sin(2t)$ .

$\theta(0) = A = \frac{3}{10}$ .  $\theta'(0) = 2B = -\frac{4}{5}$ . Hence,  $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t) = r\cos(2t - \alpha)$  where  $r(\cos\alpha, \sin\alpha) = (A, B)$ .

In particular, the amplitude is  $\sqrt{A^2 + B^2} = \sqrt{\frac{9}{100} + \frac{16}{100}} = \frac{1}{2}$ .

- (b) The characteristic equation  $r^2 + cr + 5 = 0$  has roots  $\frac{-c \pm \sqrt{c^2 - 20}}{2}$ . The system is underdamped if the solutions involve oscillations, which happens if and only if the discriminant  $\Delta = c^2 - 20$  is negative.  $c^2 - 20 < 0$  if  $c < \sqrt{20}$ . So, the system is underdamped for  $c \in (0, 2\sqrt{5})$ .
- (c) The natural frequency is 2 ( $\pm 2i$  are the roots of the characteristic equation). Hence, there will be resonance if  $\omega = 2$ .
- (d) The “old” roots are  $-1, -1$ . The “new” roots are  $\pm 2i$ . Since they don’t overlap,  $x_{sp}$  has the form  $x_{sp} = A\cos(2t) + B\sin(2t)$ .

We plug into the DE to find  $x''_{sp} + 2x'_{sp} + x_{sp} = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \stackrel{!}{=} 25\cos(2t)$ .

Solving  $-3A + 4B = 25$  and  $-3B - 4A = 0$ , we get  $B = -\frac{4}{3}A$ ,  $(-3 - \frac{16}{3})A = -\frac{25}{3}A = 25$ . So,  $A = -3$  and  $B = 4$ .

$x_{sp} = -3\cos(2t) + 4\sin(2t)$ . In particular, the amplitude is  $\sqrt{(-3)^2 + 4^2} = 5$ . □

**Problem 3.** The motion of a certain mass on a spring is described by  $x'' + x' + \frac{x}{2} = 5\sin(\omega t)$ .

- (a) Assume first that  $\omega = 1$ . Find the position function  $x(t)$  if  $x(0) = 2$  and  $x'(0) = 0$ .

- (b) Suppose the frequency of the external force is changed so that  $x'' + x' + \frac{x}{2} = 5 \sin(\omega t)$ . Does practical resonance occur for some value of  $\omega$ ? If so, for what  $\omega$ ?

**Solution.**

- (a) “Old” roots  $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$ . “New” roots  $\pm i\omega$ . Since there is no overlap,  $x_{\text{sp}}$  has the form  $x_{\text{sp}} = A_1 \cos(t) + A_2 \sin(t)$ . Plugging into DE, we find  $A_1 = -4$ ,  $A_2 = -2$ .

Hence, the general solution is  $x(t) = -4 \cos(t) - 2 \sin(t) + e^{-t/2}(c_1 \cos(t/2) + c_2 \sin(t/2))$ .

Using  $x(0) = -4 + c_1 = 2$ , we find  $c_1 = 6$ . Then  $x'(t) = 4 \sin(t) - 2 \cos(t) - \frac{1}{2}e^{-t/2}(c_1 \cos(t/2) + c_2 \sin(t/2)) + e^{-t/2}(-2 \sin(t/2) + \frac{c_2}{2} \cos(t/2))$ . Using  $x'(0) = -2 - \frac{c_1}{2} + \frac{c_2}{2} = 0$ , we also find  $c_2 = 10$ .

In summary,  $x(t) = -4 \cos(t) - 2 \sin(t) + e^{-t/2}(6 \cos(t/2) + 10 \sin(t/2))$ .

- (b) We proceed as in the first part, but now  $x_{\text{sp}}$  is of the form  $x_{\text{sp}} = A_1 \cos(\omega t) + A_2 \sin(\omega t)$ . Plugging into the DE, we find: first  $A_2 = \frac{2\omega^2 - 1}{2\omega} A_1$ , then  $A_1 = -10 \frac{2\omega}{(2\omega)^2 + (2\omega^2 - 1)^2} = -10 \frac{2\omega}{4\omega^4 + 1}$  and so  $A_2 = -10 \frac{2\omega^2 - 1}{4\omega^4 + 1}$ .

The amplitude is  $A(\omega) = \sqrt{A_1^2 + A_2^2} = \frac{10}{\sqrt{1 + 4\omega^4}}$ .

Practical resonance occurs if  $A(\omega)$  has a maximum at some  $\omega > 0$ . To investigate, we compute  $A'(\omega) = -5 \frac{16\omega^3}{(1 + 4\omega^4)^{3/2}}$ . We see that  $A'(\omega) = 0$  only for  $\omega = 0$ . Hence, there is no practical resonance here.  $\square$

**Problem 4.** Find the general solution of  $y'' - 4y' + 4y = 3e^{2x}$ .

**Solution.** The characteristic equation for the homogeneous DE has roots 2, 2 (“old” roots). The right-hand side solves a DE whose characteristic equation has roots 2 (“new” roots). Hence, there is a particular solution of the form  $y_p = Ax^2 e^{2x}$ .

To find  $A$ , we plug into the differential equation using  $y'_p = 2A(x + x^2)e^{2x}$  and  $y''_p = 2A(1 + 4x + 2x^2)e^{2x}$ :  
 $y''_p - 4y'_p + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \stackrel{!}{=} 3e^{2x}$ . Hence,  $A = \frac{3}{2}$ .

The general solution is  $(c_1 + c_2 x + \frac{3}{2}x^2)e^{2x}$ .  $\square$

**Problem 5.**

- (a) Consider the differential equation  $x^2 y'' - 4xy' + 6y = 0$ . Find all solutions of the form  $y(x) = x^r$ .  
 (b) Show that the solutions you found are independent.  
 (c) Note that the Wronskian of your solutions is zero for  $x = 0$ . Why does this not contradict the independence?  
 (d) Find the general solution of  $x^2 y'' - 4xy' + 6y = x^3$ .

**Solution.**

- (a) Plugging  $y = x^r$  into the DE and assuming  $r \geq 2$ , we get  $x^2 r(r-1)x^{r-2} - 4xr x^{r-1} + 6x^r = [r(r-1) - 4r + 6]x^r = 0$ .  $r(r-1) - 4r + 6 = (r-2)(r-3)$ . Hence, we have found the solutions  $x^2$  and  $x^3$ . Since this is a second-order equation and our solutions are independent (as we will certify next), there can be no other solutions (and we can safely ignore the case  $r < 2$ ).

- (b) The Wronskian of  $y_1 = x^2$ ,  $y_2 = x^3$  is  $W(x) = \det \begin{pmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{pmatrix} = x^4 \neq 0$  for  $x \neq 0$ .

- (c) Before using the Wronskian theorem, we have to put the DE into the form  $y'' - 4x^{-1}y' + 6x^{-2}y = 0$ . The coefficients are not defined, and hence not continuous, for  $x = 0$ . We therefore cannot apply the Wronskian criterion at  $x = 0$ .

- (d) Put DE in the form  $y'' - 4x^{-1}y' + 6x^{-2}y = x$ . The method of variation of constants shows that a particular solution is given by

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx = -x^2 \int 1 dx + x^3 \int \frac{1}{x} dx = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is  $c_1x^2 + (c_2 + \ln|x|)x^3$ . □

**Problem 6.** Solve  $\mathbf{x}' = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} \mathbf{x}$ ,  $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$ .

**Solution.** The characteristic polynomial is

$$\det \begin{pmatrix} 3-\lambda & -2 & 0 \\ -1 & 3-\lambda & -2 \\ 0 & -1 & 3-\lambda \end{pmatrix} = (3-\lambda) \det \begin{pmatrix} 3-\lambda & -2 \\ -1 & 3-\lambda \end{pmatrix} + 2 \det \begin{pmatrix} -1 & -2 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^3 - 2(3-\lambda) - 2(3-\lambda) \\ = (3-\lambda)[(3-\lambda)^2 - 4],$$

which has roots  $\lambda = 3, 3 \pm 2 = 1, 3, 5$ . These are the eigenvalues.

$$\lambda = 1. \quad \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \mathbf{v} = 0. \quad \text{We find } \mathbf{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

$$\lambda = 3. \quad \begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \mathbf{v} = 0. \quad \text{We find } \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

$$\lambda = 5. \quad \begin{pmatrix} -2 & -2 & 0 \\ -1 & -2 & -2 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{v} = 0. \quad \text{We find } \mathbf{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Consequently, the general solution is  $\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{3t} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{5t}$ .

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}.$$

$$\text{We eliminate: } \begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array} \implies \begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & 0 & 6 \end{array}. \quad \text{Hence, } c_2 = -3, c_3 = 1, c_1 = 2.$$

The IVP is solved by  $2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} e^t - 3 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} e^{5t}$ . □

# Midterm #2

MATH 286 — Differential Equations Plus  
Thursday, March 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

**Good luck!**

**Problem 1. (5 points)** Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose  $y(x) = x^2 e^{2x} \cos(x)$  is a solution. Write down the general solution.

$$y(x) =$$

**Problem 2. (20 points)** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ .

$$\mathbf{x}(t) =$$

**Problem 3. (10 points)** The position  $x(t)$  of a certain mass on a spring is described by  $x'' + cx' + 5x = F \sin(\omega t)$ .

- (a) Assume first that there is no external force, i.e.  $F = 0$ . For which values of  $c$  is the system overdamped?
- (b) Now,  $F \neq 0$  and the system is undamped, i.e.  $c = 0$ . For which values of  $\omega$ , if any, does resonance occur?

Overdamped for:	Resonance for:

**Problem 4. (20 points)** Find the general solution of the differential equation  $y^{(3)} - y = e^x + 7$ .

$y(x) =$

**Problem 5. (20 points)** Consider, for  $x > 0$ , the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right)y' + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = 0.$$

- (a) Show that the functions  $y_1(x) = x$  and  $y_2(x) = x e^x$  are solutions to this differential equation.
- (b) Using the Wronskian, show that  $y_1$  and  $y_2$  are linearly independent solutions to the above differential equation.
- (c) Find, for  $x > 0$ , the general solution to the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right)y' + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = 2x.$$

$y(x) =$

**Problem 6. (20 points)** The motion of a certain mass on a spring is described by  $x'' + 2x' + 2x = 5 \sin(t)$ .

- (a) What is the amplitude of the resulting steady periodic oscillations?
- (b) Assume that the mass is initially at rest (i.e.  $x(0) = 0$ ,  $x'(0) = 0$ ) and find the position function  $x(t)$ .

Amplitude:	$x(t) =$

**Problem 7. (5 points)** Let  $y_p$  be any solution to the inhomogeneous linear differential equation  $y'' + xy = e^x$ . Find a homogeneous linear differential equation which  $y_p$  solves. *Hint:* Do not attempt to solve the DE.

Homogeneous linear DE:

# Midterm #2

MATH 286 — Differential Equations Plus  
Thursday, March 13

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

**Good luck!**

**Problem 1. (5 points)** Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose  $y(x) = x^2 e^{2x} \cos(x)$  is a solution. Write down the general solution.

**Solution.** The general solution is  $(c_1 + c_2 x + c_3 x^2) e^{2x} \cos(x) + (c_4 + c_5 x + c_6 x^2) e^{2x} \sin(x)$ . □

**Problem 2. (20 points)** Find the general solution of  $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ .

**Solution.** The characteristic polynomial  $(1 - \lambda)^2 - 4$  has roots  $\lambda = -1, 3$ .

For  $\lambda = 3$ , solving  $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , we find the eigenvector  $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

For  $\lambda = -1$ , solving  $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ , we find the eigenvector  $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

Hence, the general solution is  $\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$ . □



**Problem 3. (10 points)** The position  $x(t)$  of a certain mass on a spring is described by  $x'' + cx' + 5x = F \sin(\omega t)$ .

(a) Assume first that there is no external force, i.e.  $F=0$ . For which values of  $c$  is the system overdamped?

(b) Now,  $F \neq 0$  and the system is undamped, i.e.  $c=0$ . For which values of  $\omega$ , if any, does resonance occur?

**Solution.**

(a) The discriminant of the characteristic equation is  $c^2 - 20$ . Hence the system is overdamped if  $c^2 - 20 > 0$ , that is  $c > \sqrt{20} = 2\sqrt{5}$ .

(b) The natural frequency is  $\sqrt{5}$ . Resonance therefore occurs if  $\omega = \sqrt{5}$ . □

**Problem 4. (20 points)** Find the general solution of the differential equation  $y^{(3)} - y = e^x + 7$ .

**Solution.** Let us first solve the homogeneous equation  $y''' - y = 0$ . Its characteristic polynomial  $r^3 - 1 = (r - 1)(r^2 + r + 1)$  has roots  $r = 1$  and  $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ .

There is a particular solution of the form  $y_p = Axe^x + B$ .

$$y'_p = A(x+1)e^x, \quad y''_p = A(x+2)e^x, \quad y'''_p = A(x+3)e^x$$

Plugging into the DE, we get  $y'''_p - y_p = 3Ae^x - B \stackrel{!}{=} e^x + 7$ . Consequently,  $A = \frac{1}{3}$ ,  $B = -7$ .

Hence, the general solution is  $-7 + (c_1 + \frac{1}{3}x)e^x + c_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$ . □

**Problem 5. (20 points)** Consider, for  $x > 0$ , the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 0.$$

- (a) Show that the functions  $y_1(x) = x$  and  $y_2(x) = x e^x$  are solutions to this differential equation.
- (b) Using the Wronskian, show that  $y_1$  and  $y_2$  are linearly independent solutions to the above differential equation.
- (c) Find, for  $x > 0$ , the general solution to the second-order differential equation

$$y'' - \left(1 + \frac{2}{x}\right) y' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y = 2x.$$

**Solution.**

- (a) We have

$$y_1'' - \left(1 + \frac{2}{x}\right) y_1' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_1 = 0 - \left(1 + \frac{2}{x}\right) + \left(\frac{1}{x} + \frac{2}{x^2}\right) x = 0,$$

and

$$y_2'' - \left(1 + \frac{2}{x}\right) y_2' + \left(\frac{1}{x} + \frac{2}{x^2}\right) y_2 = x e^x + 2e^x - \left(1 + \frac{2}{x}\right) (x e^x + e^x) + \left(\frac{1}{x} + \frac{2}{x^2}\right) (x e^x) = 0.$$

- (b) The Wronskian of  $y_1$  and  $y_2$  is given by

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = \det \begin{bmatrix} x & x e^x \\ 1 & x e^x + e^x \end{bmatrix} = x^2 e^x.$$

Since  $W(x) \neq 0$  on the domain of definition for the differential equation, the Wronskian theorem implies that  $y_1$  and  $y_2$  are linearly independent.

- (c) The general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x) = c_1 x + c_2 x e^x + y_p(x),$$

where  $y_p$  is any particular solution to the above non-homogeneous equation. To find such a  $y_p$ , we will use the variation of parameters formula:

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) = u_1(x) x + u_2(x) x e^x,$$

where

$$\begin{aligned} u_1(x) &= - \int \frac{2x y_2(x)}{W(x)} dx = - \int 2 dx = -2x, \\ u_2(x) &= \int \frac{2x y_1(x)}{W(x)} dx = \int 2 e^{-x} dx = -2 e^{-x}. \end{aligned}$$

So,

$$y(x) = c_1 x + c_2 x e^x - 2x^2 - 2x = d_1 x + d_2 x e^x - 2x^2.$$

□

**Problem 6. (20 points)** The motion of a certain mass on a spring is described by  $x'' + 2x' + 2x = 5 \sin(t)$ .

- (a) What is the amplitude of the resulting steady periodic oscillations?
- (b) Assume that the mass is initially at rest (i.e.  $x(0) = 0$ ,  $x'(0) = 0$ ) and find the position function  $x(t)$ .

**Solution.**

- (a) The characteristic polynomial of the associated homogeneous DE has roots  $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ .

Hence,  $x_{\text{sp}}$  is the form  $x_{\text{sp}} = A_1 \cos(t) + A_2 \sin(t)$ .

We compute  $x'_{\text{sp}} = -A_1 \sin(t) + A_2 \cos(t)$  and  $x''_{\text{sp}} = -A_1 \cos(t) - A_2 \sin(t)$ .

Plugging into the DE gives  $x''_{\text{sp}} + 2x'_{\text{sp}} + 2x_{\text{sp}} = (A_1 + 2A_2)\cos(t) + (A_2 - 2A_1)\sin(t) \stackrel{!}{=} 5\sin(t)$ . Consequently,  $A_1 + 2A_2 = 0$  and  $A_2 - 2A_1 = 5$ , resulting in  $A_1 = -2$ ,  $A_2 = 1$ .

Thus,  $x_{\text{sp}} = -2\cos(t) + \sin(t)$ . The amplitude is  $\sqrt{(-2)^2 + 1^2} = \sqrt{5}$ .

- (b) From first part, we know that  $x(t) = -2\cos(t) + \sin(t) + e^{-t}(c_1 \cos(t) + c_2 \sin(t))$ .

Using  $x(0) = -2 + c_1 = 0$  we find  $c_1 = 2$ .

$x'(t) = 2\sin(t) + \cos(t) - e^{-t}(2\cos(t) + c_2 \sin(t)) + e^{-t}(-2\sin(t) + c_2 \cos(t))$ . Hence,  $x'(0) = 1 - 2 + c_2 = 0$  results in  $c_2 = 1$ .

In conclusion,  $x(t) = -2\cos(t) + \sin(t) + e^{-t}(2\cos(t) + \sin(t))$ . □

**Problem 7. (5 points)** Let  $y_p$  be any solution to the inhomogeneous linear differential equation  $y'' + xy = e^x$ . Find a homogeneous linear differential equation which  $y_p$  solves. *Hint:* Do not attempt to solve the DE.

**Solution.** Apply  $D = \frac{d}{dx}$  to both sides of the differential equation to get  $y''' + xy' + y = e^x$ . Subtracting the two differential equations, we get the homogeneous linear DE  $y''' - y'' + xy' + (1-x)y = 0$ .

[Note that this is the same as applying  $D - 1$  to both sides of the DE, in analogy with our approach for solving inhomogeneous linear DEs with constant coefficients.] □

Student Name: \_\_\_\_\_  
Student Net ID: \_\_\_\_\_

MATH 286 SECTION G1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 2

October 18, 2012

INSTRUCTOR: M. BRANNAN

**INSTRUCTIONS**

- This exam 50 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **EXPLAIN YOUR WORK!** Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- **GOOD LUCK!**

**PLEASE NOTE:** “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

Question:	1	2	3	4	Total
Points:	15	15	12	12	54
Score:					

1. Consider the ordinary differential equation

$$Ly = y^{(3)} - 2y'' + 2y' = x + xe^x.$$

- (a) (5 points) Determine the complementary function  $y_c(x)$  for the above ODE.

**Solution:** The characteristic polynomial is

$$P(r) = r^3 - 2r^2 + 2r = r(r^2 - 2r + 2) = r(r - 1 - i)(r - 1 + i).$$

Therefore the complementary function (which is the solution to  $Ly = 0$ ) is

$$y_c(x) = c_1 + c_2 e^x \cos x + c_3 e^x \sin x.$$

- (b) (8 points) Find a particular solution  $y_p(x)$  to the above ODE.

**Solution:** Since  $F(x) = x + xe^x$ , a first guess at a trial solution would be of the form  $Ax + B + (Cx + D)e^x$ . But there is overlap with the complementary function  $y_c(x)$  coming from the constant  $B$ . To eliminate this we multiply  $Ax + B$  by  $x$  and get a particular solution of the form

$$\begin{aligned} y_p(x) &= x(Ax + B) + (Cx + D)e^x \\ \implies y'_p(x) &= 2Ax + B + Ce^x + Cxe^x + De^x = 2Ax + B + Cxe^x + (C + D)e^x \\ \implies y''_p(x) &= 2A + (C + D)e^x + Ce^x + Cxe^x = 2A + Cxe^x + (2C + D)e^x \\ \implies y^{(3)}_p(x) &= (2C + D)e^x + Ce^x + Cxe^x = Cxe^x + (3C + D)e^x. \end{aligned}$$

Setting  $Ly_p = F$  and comparing coefficients, we get

$$(\text{coefficient of } xe^x) : 1 = C - 2C + 2C \implies C = 1$$

$$(\text{coefficient of } e^x) : 0 = 3C + D - 2(2C + D) + 2(C + D) \implies D = -1$$

$$(\text{coefficient of } x) : 1 = 4A \implies A = \frac{1}{4}$$

$$(\text{coefficient of } 1) : 0 = -4A + 2B \implies B = \frac{1}{2}.$$

- (c) (2 points) Write down the general solution to the above ODE.

**Solution:** The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1 + c_2 e^x \cos x + c_2 e^x \sin x + \frac{1}{4}x^2 + \frac{1}{2}x + (x-1)e^x$$

2. Consider the ordinary differential equation

$$Ly = y'' - \frac{1}{x}y' + \frac{1}{x^2}y = x^3.$$

- (a) (4 points) Verify that  $y_1(x) = x$  and  $y_2(x) = x \ln |x|$  are solutions to the associated homogeneous ODE

$$Ly = 0 \quad (x \neq 0).$$

**Solution:**

$$\begin{aligned} y_1'' - \frac{1}{x}y_1' + \frac{1}{x^2}y_1 &= 0 - \frac{1}{x} + \frac{1}{x} = 0. \\ y_2'' - \frac{1}{x}y_2' + \frac{1}{x^2}y_2 &= \frac{1}{x} - \frac{1}{x}(\ln |x| + 1) + \frac{1}{x} \ln |x| = 0. \end{aligned}$$

- (b) (5 points) Compute the Wronskian  $W(y_1, y_2)$  for the pair of functions  $y_1, y_2$  above. Are  $y_1$  and  $y_2$  linearly independent on the interval  $I = (0, \infty)$ ? Why or why not?

**Solution:**

$$W(x) = \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = x(\ln |x| + 1) - 1(x \ln |x|) = x.$$

Since  $y_1, y_2$  are solutions to the ODE  $y'' + p(x)y' + q(x) = 0$  and  $W(y_1, y_2) \neq 0$  on  $I$ , the theorem on linear independence and Wronskians implies that these functions are linearly independent on  $I$ .

- (c) (6 points) Find a particular solution to

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = x^3 \quad (x > 0).$$

(**Hint:** One possible approach is to use the variation of parameters method.)

**Solution:** Let

$$u_1(x) = - \int \frac{y_2(x)x^3}{W(x)} dx = - \int x^3 \ln |x| dx = \frac{-x^4}{4} \ln |x| + \int \frac{x^3}{4} dx = \frac{-x^4}{4} \ln |x| + \frac{x^4}{16}.$$

$$u_2(x) = \int \frac{y_1(x)x^3}{W(x)} dx = \int x^3 dx = \frac{x^4}{4}.$$

Then the method of variation of parameters gives

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{x^5}{4} \left( \frac{1}{4} - \ln |x| \right) + \frac{x^5}{4} \ln |x| = \frac{x^5}{16}.$$

3. (12 points) For the following endpoint problem, determine all eigenvalues  $\lambda \in \mathbb{R}$  and their associated eigenfunctions:

$$y'' - 4y' + \lambda y = 0, \quad y(0) = y(1) = 0.$$

**Solution:** The characteristic equation is

$$0 = P(r) = r^2 - 4r + \lambda \iff r = 2 \pm \frac{\sqrt{16 - 4\lambda}}{2} = 2 \pm \sqrt{4 - \lambda}.$$

**Case 1:** If  $\lambda < 4$ , the general solution is

$$y(t) = c_1 e^{(2+\sqrt{4-\lambda})t} + c_2 e^{(2-\sqrt{4-\lambda})t}.$$

Plugging in the endpoint conditions, we find  $c_1 = c_2 = 0$ . So, there are no eigenvalues  $\lambda < 4$ .

**Case 2:** If  $\lambda = 4$ , the general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}.$$

Plugging in the endpoint conditions, we find

$$0 = y(0) = c_1 \quad 0 = y(1) = c_2 e^2 \implies c_1 = c_2 = 0.$$

So  $\lambda = 4$  is not an eigenvalue.

**Case 3:** If  $\lambda > 4$ , the general solution is

$$y(t) = c_1 e^{2t} \cos \sqrt{\lambda - 4}t + c_2 e^{2t} \sin \sqrt{\lambda - 4}t.$$

The endpoint conditions then give

$$0 = y(0) = c_1 \quad 0 = y(1) = c_2 e^2 \sin \sqrt{\lambda - 4} \implies \sqrt{\lambda - 4} = n\pi \quad (n = 1, 2, 3, \dots).$$

So the eigenvalues are

$$\lambda_n = 4 + n^2 \pi^2 \quad (n = 1, 2, \dots),$$

and the eigenfunctions are

$$y_n(t) = e^{2t} \sin(n\pi t)$$



4. Consider a mass-spring-dashpot system with mass  $m = 1\text{kg}$ , spring constant  $k = 4\text{ N/m}$  and dashpot damping constant  $\beta > 0\text{ N s/m}$ . Let  $x(t)$  denote the displacement (in metres, at time  $t$ ) of the mass from its equilibrium resting position.

- (a) (4 points) For what values of  $\beta$  is the system underdamped?

**Solution:** The characteristic polynomial for this system is

$$P(r) = mr^2 + \beta r + k = r^2 + \beta r + 4,$$

which has roots  $r = \frac{-\beta}{2} \pm \frac{\sqrt{\beta^2 - 16}}{2}$ . The system is underdamped when the roots are complex. I.e., when  $\beta^2 - 16 < 0 \iff 0 < \beta < 4$ .

For the remainder of the problem, assume that the dashpot is **disconnected** from the system (i.e., set  $\beta = 0$ ) and that an external force  $F(t) = 2\sin\omega t$  Newtons is applied to the mass.

- (b) (2 points) At what forcing frequency  $\omega$  will resonance occur in the forced system?

**Solution:** Resonance will occur when  $\omega = \omega_0 = \sqrt{k/m} = 2$ , which is the natural frequency of the system.

- (c) (6 points) Write down the general solution  $x(t)$  in this case.

**Solution:** The complementary function for this equation is

$$x_c(t) = C \cos(2t - \alpha).$$

A particular solution will be of the form  $x_p(t) = At \cos(2t) + Bt \sin(2t)$ , setting  $y_p'' + 4y_p = 2\sin(2t)$ , we get

$$2\sin(2t) = -4A \sin(2t) - 4At \cos(2t) + 4B \cos(2t) - 4Bt \sin(2t) + 4At \cos(2t) + 4Bt \sin(2t),$$

giving  $A = -1/2$  and  $B = 0$ . Thus the general solution is

$$x(t) = x_c(t) + x_p(t) = C \cos(2t - \alpha) - \frac{t}{2} \cos(2t).$$

*(Extra work space.)*

# Preparing Midterm #3

MATH 286 — Differential Equations Plus

April 17, 2014

*The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka!" but "That's funny ...".*

— Isaac Asimov (1920–1992) —

**Problem 1.** Let  $A$  be a  $2 \times 2$  matrix such that  $e^{At} = \begin{pmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (c+t)e^{rt} \end{pmatrix}$ . What are the values of  $c$  and  $r$ ?

**Problem 2.** Consider  $\mathbf{x}' = A\mathbf{x}$  where  $A = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$ .

- (a) Find the general solution.
- (b) Find  $e^{At}$ .
- (c) Find a particular solution to  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}$ .

**Problem 3.** The mixtures in three tanks  $T_1, T_2, T_3$  are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at 15 gal/min. Mixed solution from  $T_1$  is pumped into  $T_2$  at 10 gal/min and from  $T_2$  into  $T_3$  at 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by  $x_i(t)$  the amount (in pounds) of salt in tank  $T_i$  at time  $t$  (in minutes). Derive a system of linear differential equations for the  $x_i$ .

**Problem 4.** Let  $A$  be a  $3 \times 3$  matrix such that  $e^{At} = \begin{pmatrix} e^{2t} - te^{-t} & te^{-t} & -e^{2t} + (t+1)e^{-t} \\ e^{2t} - e^{-t} & e^{-t} & -e^{2t} + e^{-t} \\ -te^{-t} & te^{-t} & (t+1)e^{-t} \end{pmatrix}$ .

- (a) What are the eigenvalues of  $A$ ? Indicate if an eigenvalue is repeated and what its defect is.
- (b) Solve the initial value problem  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = (1 \ 0 \ 1)^T$ .
- (c) Find a particular solution to  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ .
- (d) Find  $A$ . Also, as a challenge, find  $A^{100}$ .

**Problem 5.** The  $6 \times 6$  matrix  $A$  has eigenvalues  $-3, -3, 0, 1, 1, 1$ .

- (a) Which eigenvalues can be defective? Briefly describe in *all* possible scenarios what sort of (generalized) eigenvectors would arise, and what form the solutions take in each case.
- (b) We wish to solve  $\mathbf{x}' = A\mathbf{x} + (2t^2, e^{-2t}\sin(t), 0, -1, 0, t\cos(t))^T$ . Write down a particular solution  $\mathbf{x}_p$  with undetermined coefficients. It should have as few terms as possible and still work for any matrix  $A$  with the stated eigenvalues.

**Problem 6.** Consider  $\mathbf{x}' = A\mathbf{x}$  where  $A = \begin{pmatrix} 2 & 4 & -1 \\ 7 & -1 & -5 \\ -1 & 1 & -1 \end{pmatrix}$ .

(a) Find a fundamental matrix.

*Hint:* The eigenvalues of  $A$  are  $-3, -3, 6$ .

(b) Solve the initial value problem with  $\mathbf{x}(0) = (3 \ 0 \ 0)^T$ .

**Problem 7.** Let  $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ .

(a) Show that the matrix  $N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  is nilpotent.

(b) Use the fact that  $N$  is nilpotent, to find  $e^{At}$ .

(c) Solve the initial value problem  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = (1 \ 2 \ 3)^T$ .

(d) Find a particular solution of  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix}$ .

(e) Use a different method to solve the previous problem.

**Problem 8.** Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

You may use that the characteristic polynomial has the repeated roots  $1 \pm i$ . The general solution should be given in terms of real-valued functions.

*The most exciting phrase to hear in science, the one that heralds new discoveries, is not “Eureka!” but “That’s funny ...”.*

— Isaac Asimov (1920–1992) —

**Problem 1.** Let  $A$  be a  $2 \times 2$  matrix such that  $e^{At} = \begin{pmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (c+t)e^{rt} \end{pmatrix}$ . What are the values of  $c$  and  $r$ ?

**Solution.** Using that  $e^{At}|_{t=0} = I$ , we conclude that  $c = 1$ . The presence of the term  $te^{2t}$  shows that 2 is a repeated eigenvalue. Since  $A$  is  $2 \times 2$  and therefore has exactly 2 eigenvalues (counting with multiplicity), it follows that  $r = 2$  as well.  $\square$

**Problem 2.** Consider  $\mathbf{x}' = A\mathbf{x}$  where  $A = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$ .

(a) Find the general solution.

(b) Find  $e^{At}$ .

(c) Find a particular solution to  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}$ .

**Solution.**

(a) The eigenvalues of  $A$  are 0, 1 with eigenvectors  $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  and  $\mathbf{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ . Hence the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^t.$$

(b) From the first part, we know that a fundamental matrix is given by  $\Phi(t) = \begin{pmatrix} 1 & e^t \\ 2 & 3e^t \end{pmatrix}$ . Then

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} 1 & e^t \\ 2 & 3e^t \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 - 2e^t & -1 + e^t \\ 6 - 6e^t & -2 + 3e^t \end{pmatrix}.$$

(c) Using variation of constants,

$$\begin{aligned} \mathbf{x}(t) &= e^{At} \int e^{-At} \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix} dt \\ &= e^{At} \int \begin{pmatrix} 3 - 2e^{-t} & -1 + e^{-t} \\ 6 - 6e^{-t} & -2 + 3e^{-t} \end{pmatrix} \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix} dt \\ &= e^{At} \int \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix} dt = e^{At} \begin{pmatrix} -1/t \\ -2/t \end{pmatrix} \\ &= \begin{pmatrix} 3 - 2e^t & -1 + e^t \\ 6 - 6e^t & -2 + 3e^t \end{pmatrix} \begin{pmatrix} -1/t \\ -2/t \end{pmatrix} = \begin{pmatrix} -1/t \\ -2/t \end{pmatrix}. \end{aligned}$$

□

**Problem 3.** The mixtures in three tanks  $T_1, T_2, T_3$  are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at 15 gal/min. Mixed solution from  $T_1$  is pumped into  $T_2$  at 10 gal/min and from  $T_2$  into  $T_3$  at 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by  $x_i(t)$  the amount (in pounds) of salt in tank  $T_i$  at time  $t$  (in minutes). Derive a system of linear differential equations for the  $x_i$ .

**Solution.** Note that at time  $t$ ,  $T_1$  contains  $10 + 5t$  gal of solution. Likewise,  $T_2$  contains 10 gal, and  $T_3$   $10 + 10t$ .

Consider a short interval of time  $(t, t + \Delta t)$ .

$$\begin{aligned}\Delta x_1 &\approx 15 \cdot 2 \cdot \Delta t - 10 \cdot \frac{x_1}{10 + 5t} \cdot \Delta t &\implies x_1' &= 30 - \frac{2x_1}{2+t} \\ \Delta x_2 &\approx 10 \cdot \frac{x_1}{10 + 5t} \cdot \Delta t - 10 \cdot \frac{x_2}{10} \cdot \Delta t &\implies x_2' &= \frac{2x_1}{2+t} - x_2 \\ \Delta x_3 &\approx 10 \cdot \frac{x_2}{10} \cdot \Delta t &\implies x_3' &= x_2\end{aligned}$$

We also have the initial conditions  $x_1(0) = 0$ ,  $x_2(0) = 0$ ,  $x_3(0) = 0$ . In matrix form, writing  $\mathbf{x} = (x_1, x_2, x_3)$ , this is

$$\mathbf{x}' = \begin{pmatrix} -\frac{2}{2+t} & 0 & 0 \\ \frac{2}{2+t} & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients (which means that we cannot apply our knowledge of eigenvectors to solve the complementary solution). □

**Problem 4.** Let  $A$  be a  $3 \times 3$  matrix such that  $e^{At} = \begin{pmatrix} e^{2t} - te^{-t} & te^{-t} & -e^{2t} + (t+1)e^{-t} \\ e^{2t} - e^{-t} & e^{-t} & -e^{2t} + e^{-t} \\ -te^{-t} & te^{-t} & (t+1)e^{-t} \end{pmatrix}$ .

- What are the eigenvalues of  $A$ ? Indicate if an eigenvalue is repeated and what its defect is.
- Solve the initial value problem  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = (1 \ 0 \ 1)^T$ .
- Find a particular solution to  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ .
- Find  $A$ . Also, as a challenge, find  $A^{100}$ .

**Solution.**

- The eigenvalues are 2,  $-1$ ,  $-1$ . The eigenvalue  $-1$  is repeated and has defect 1.

$$(b) \quad \mathbf{x}(t) = e^{At} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} - te^{-t} \\ e^{2t} - e^{-t} \\ -te^{-t} \end{pmatrix} + \begin{pmatrix} -e^{2t} + (t+1)e^{-t} \\ -e^{2t} + e^{-t} \\ (t+1)e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t} \\ 0 \\ e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$(c) \quad \mathbf{x}_p(t) = e^{At} \int e^{-At} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} dt = e^{At} \int \begin{pmatrix} e^{-2t} \\ e^{-2t} \\ 0 \end{pmatrix} dt = -\frac{1}{2} e^{-2t} e^{At} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -\frac{1}{2} e^{-2t} \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

- (d) Recall that  $\frac{d}{dt}e^{At} = Ae^{At}$ . Setting  $t=0$  then gives  $A = \begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & -3 \\ -1 & 1 & 0 \end{pmatrix}$ .

Our strategy to find  $A^{100}$  is to use  $\frac{d^{100}}{dt^{100}}e^{At} = A^{100}e^{At}$ . Clearly,  $\frac{d^{100}}{dt^{100}}e^{2t} = 2^{100}e^{2t}$  and  $\frac{d^{100}}{dt^{100}}e^{-t} = e^{-t}$ . Note that  $\frac{d}{dt}(te^{-t}) = (-t+1)e^{-t}$ ,  $\frac{d^2}{dt^2}(te^{-t}) = (t-2)e^{-t}$ ,  $\frac{d^3}{dt^3}(te^{-t}) = (-t+3)e^{-t}$ . Continuing, we find that  $\frac{d^n}{dt^n}(te^{-t}) = (-1)^n(t-n)e^{-t}$ . In particular,  $\frac{d^{100}}{dt^{100}}(te^{-t}) = (t-100)e^{-t}$ . Therefore,

$$\frac{d^{100}}{dt^{100}}e^{At} = \begin{pmatrix} 2^{100}e^{2t} - (t-100)e^{-t} & (t-100)e^{-t} & -2^{100}e^{2t} + e^{-t} + (t-100)e^{-t} \\ 2^{100}e^{2t} - e^{-t} & e^{-t} & -2^{100}e^{2t} + e^{-t} \\ -(t-100)e^{-t} & (t-100)e^{-t} & e^{-t} + (t-100)e^{-t} \end{pmatrix}.$$

Setting  $t=0$ , we find

$$A^{100} = \begin{pmatrix} 2^{100} + 100 & -100 & -2^{100} - 99 \\ 2^{100} - 1 & 1 & -2^{100} + 1 \\ 100 & -100 & -99 \end{pmatrix}.$$

You should feel rightfully proud of your new powers! As you just discovered, the matrix exponential makes it possible to easily computer any power of a matrix (ours was a hard case because of the terms  $te^{\lambda t}$  due to a defective eigenvalue; usually taking derivatives requires no thinking).  $\square$

**Problem 5.** The  $6 \times 6$  matrix  $A$  has eigenvalues  $-3, -3, 0, 1, 1, 1$ .

- Which eigenvalues can be defective? Briefly describe in *all* possible scenarios what sort of (generalized) eigenvectors would arise, and what form the solutions take in each case.
- We wish to solve  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 2t^2, e^{-2t}\sin(t), 0, -1, 0, t\cos(t) \end{pmatrix}^T$ . Write down a particular solution  $\mathbf{x}_p$  with undetermined coefficients. It should have as few terms as possible and still work for any matrix  $A$  with the stated eigenvalues.

**Solution.**

- The eigenvalues  $\lambda = -3$  and  $\lambda = 1$  might be defective.  $\lambda = -3$  may have no defect or defect 1.  $\lambda = 1$  may have no defect or defect 1 or defect 2. We describe the possibilities individually.

**$\lambda = -3$  no defect.** In this case, we find two (independent) eigenvectors for  $\lambda = -3$ .

**$\lambda = -3$  defect 1.** There is one chain  $\mathbf{v}_1, \mathbf{v}_2$  of generalized eigenvectors for  $\lambda = -3$ .

**$\lambda = 1$  no defect.** Three (independent) eigenvectors for  $\lambda = 1$ .

**$\lambda = 1$  defect 1.** There is one chain  $\mathbf{v}_1, \mathbf{v}_2$  of generalized eigenvectors for  $\lambda = 1$ , as well as a second (independent) eigenvector  $\mathbf{w}$  for  $\lambda = 1$ .

**$\lambda = 1$  defect 2.** One chain  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  of generalized eigenvectors for  $\lambda = 1$ .

- There is a particular solution  $\mathbf{x}_p$  of the form

$$\mathbf{x}_p(t) = (\mathbf{a}_1t^3 + \mathbf{a}_2t^2 + \mathbf{a}_3t + \mathbf{a}_4) + \mathbf{b}_1e^{-2t}\sin(t) + \mathbf{b}_2e^{-2t}\cos(t) + (\mathbf{c}_1t + \mathbf{c}_2)\cos(t) + (\mathbf{d}_1t + \mathbf{d}_2)\sin(t).$$

Note that this is  $10 \times 6 = 60$  undeterminates. Let's keep our theoretical attitude...  $\square$

**Problem 6.** Consider  $\mathbf{x}' = A\mathbf{x}$  where  $A = \begin{pmatrix} 2 & 4 & -1 \\ 7 & -1 & -5 \\ -1 & 1 & -1 \end{pmatrix}$ .

(a) Find a fundamental matrix.

*Hint:* The eigenvalues of  $A$  are  $-3, -3, 6$ .

(b) Solve the initial value problem with  $\mathbf{x}(0) = (3 \ 0 \ 0)^T$ .

**Solution.**

(a) The eigenvalues of  $A$  are  $-3, -3, 6$ .

For  $\lambda = 6$  we find the eigenvector  $\mathbf{v} = (1 \ 1 \ 0)^T$ .

For  $\lambda = -3$  we find the eigenvector  $\mathbf{w}_1 = (1 \ -1 \ 1)^T$  but no second one.  $\lambda = -3$  thus has defect 1.

Therefore there has to be a chain of length 2.

We solve  $\begin{pmatrix} 5 & 4 & -1 \\ 7 & 2 & -5 \\ -1 & 1 & 2 \end{pmatrix} \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and find, for instance,  $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1/3 \\ 1/3 \end{pmatrix}$ .

Fundamental matrix:  $\Phi(t) = \begin{pmatrix} e^{6t} & e^{-3t} & te^{-3t} \\ e^{6t} & -e^{-3t} & (1/3 - t)e^{-3t} \\ 0 & e^{-3t} & (1/3 + t)e^{-3t} \end{pmatrix}$ .

(b) We need to find  $\mathbf{c}$  such that  $\mathbf{x}(t) = \Phi(t)\mathbf{c}$  satisfies  $\mathbf{x}(0) = (3 \ 0 \ 0)^T$ .

$\mathbf{x}(0) = \Phi(0)\mathbf{c} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1/3 \\ 0 & 1 & 1/3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ . We solve and find  $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ .

Hence  $\mathbf{x}(t) = \begin{pmatrix} 2e^{6t} + (1 - 3t)e^{-3t} \\ 2e^{6t} - (2 - 3t)e^{-3t} \\ -3te^{-3t} \end{pmatrix}$ . □

**Problem 7.** Let  $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ .

(a) Show that the matrix  $N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  is nilpotent.

(b) Use the fact that  $N$  is nilpotent, to find  $e^{At}$ .

(c) Solve the initial value problem  $\mathbf{x}' = A\mathbf{x}$ ,  $\mathbf{x}(0) = (1 \ 2 \ 3)^T$ .

(d) Find a particular solution of  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix}$ .

(e) Use a different method to solve the previous problem.

**Solution.**

(a) We compute  $N^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $N^3 = 0$ .

(b) Note that  $At = 2It + Nt$ . Since the matrices  $2It$  and  $Nt$  commute (why?!), we have

$$\begin{aligned} e^{At} &= e^{2It}e^{Nt} = \begin{pmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{2t} \end{pmatrix} \left[ I + Nt + \frac{1}{2}(Nt)^2 \right] \\ &= e^{2t} \left[ I + \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -t^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & -t^2/2 & -t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} e^{2t}. \end{aligned}$$



(c) This is easy now!  $\mathbf{x}(t) = e^{At} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-3t-t^2 \\ 2 \\ 3+2t \end{pmatrix} e^{2t}$ .

(d) Using variation of constants, a particular solution is

$$\begin{aligned} \mathbf{x}_p(t) &= e^{At} \int e^{-At} \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix} dt = e^{At} \int \begin{pmatrix} 1 & -t^2/2 & t \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{pmatrix} e^{-2t} \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix} dt = e^{At} \int \begin{pmatrix} 1+t^2/2e^{-t} \\ -e^{-t} \\ te^{-t} \end{pmatrix} dt \\ &= \begin{pmatrix} 1 & -t^2/2 & -t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} e^{2t} \begin{pmatrix} t-t^2/2e^{-t}-te^{-t}-e^{-t} \\ e^{-t} \\ -te^{-t}-e^{-t} \end{pmatrix} = \begin{pmatrix} te^{2t}-e^t \\ e^t \\ -e^t \end{pmatrix}. \end{aligned}$$

(e) Let us use undetermined coefficients. From  $e^{At}$  we know that  $A$  has eigenvalues 2, 2, 2. Let us split the problem into two parts:  $\mathbf{x}' = A\mathbf{x} + e^t(0, -1, 0)^T$  and  $\mathbf{x}' = A\mathbf{x} + e^{2t}(1, 0, 0)^T$ .

For the first part, we look for a solution of the form  $\mathbf{x}_p = \mathbf{a}e^t$ . Plugging into the DE, we get

$$\mathbf{x}' = \mathbf{a}e^t \stackrel{!}{=} A\mathbf{x} + \mathbf{f} = A\mathbf{a}e^t + e^t \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \iff (A - I)\mathbf{a} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \iff \mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

In the other case, severe duplication occurs. We know that, in the worst case, there is a solution  $\mathbf{x}_p$  of the form  $\mathbf{x}_p = (\mathbf{a} + \mathbf{b}t + \mathbf{c}t^2 + \mathbf{d}t^3)e^{2t}$ . For practical purposes, and when working by hand, it still makes sense to first look if there exists a simpler solution  $\mathbf{x}_p = \mathbf{a}e^{2t}$  before adding a power of  $t$  each time we find no solution.

$$\mathbf{x}' = 2\mathbf{a}e^{2t} \stackrel{!}{=} A\mathbf{x} + \mathbf{f} = A\mathbf{a}e^{2t} + e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \iff (A - 2I)\mathbf{a} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{a} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \iff \mathbf{a} = \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}.$$

We are lucky and have already found a solution (we can set  $c$  to anything, for instance  $c=0$ )!

Combining, we have the particular solution (of the original problem)  $\mathbf{x}_p = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} -e^t \\ e^t \\ e^{2t}-e^t \end{pmatrix}$ .

Note that this looks rather different from the solution found in the previous problem. This is explained by

$$\begin{pmatrix} -e^t \\ e^t \\ e^{2t}-e^t \end{pmatrix} = \begin{pmatrix} te^{2t}-e^t \\ e^t \\ -e^t \end{pmatrix} + \begin{pmatrix} -te^{2t} \\ 0 \\ e^{2t} \end{pmatrix}, \quad \text{with } \begin{pmatrix} -te^{2t} \\ 0 \\ e^{2t} \end{pmatrix} = e^{At} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ a solution of } \mathbf{x}' = A\mathbf{x}.$$

□

**Problem 8.** Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

You may use that the characteristic polynomial has the repeated roots  $1 \pm i$ . The general solution should be given in terms of real-valued functions.

**Solution.** Let us find the eigenvectors for the eigenvalue  $\lambda = 1 - i$ .

$$\begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 1 & 0 & i & 1 \\ 0 & 1 & 0 & i \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array} \xrightarrow{r_3=r_3+ir_1} \begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 0 & i & 0 & 1 \\ 0 & 1 & 0 & i \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array} \xrightarrow{r_3=r_3-r_2, r_4=r_4+ir_2} \begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \end{array} \Rightarrow \mathbf{v} = \begin{pmatrix} c \\ 0 \\ ic \\ 0 \end{pmatrix}$$

Let us choose  $c = 1$ . There was only one degree of freedom, so the defect of  $\lambda$  is 1. We have to construct a chain starting with  $\mathbf{v}_1 = (1, 0, i, 0)^T$ . We extend the above elimination:

$$\begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 1 & 0 & i & 1 \\ 0 & 1 & 0 & i \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \end{array} \xrightarrow{r_3=r_3+ir_1} \begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 0 & i & 0 & 1 \\ 0 & 1 & 0 & i \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \end{array} \xrightarrow{r_3=r_3-r_2, r_4=r_4+ir_2} \begin{array}{c|c} \begin{array}{cccc} i & 1 & -1 & 0 \\ 0 & i & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \end{array} \Rightarrow \mathbf{v}_2 = \begin{pmatrix} c \\ 1 \\ ic \\ i \end{pmatrix}$$

We again choose  $c = 0$ . Summarizingly, we found the two complex solutions

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} e^{(1-i)t}, \quad \mathbf{x}_2 = \left[ \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix} \right] e^{(1-i)t} = \begin{pmatrix} t \\ 1 \\ it \\ i \end{pmatrix} e^{(1-i)t}.$$

(Together with their conjugates, we actually have four independent solutions). By taking real and imaginary parts (recall that  $e^{(1-i)t} = e^t(\cos(t) - i \sin(t))$ ), we conclude that four independent real solutions are given by

$$\operatorname{Re}(\mathbf{x}_1) = e^t \begin{pmatrix} \cos(t) \\ 0 \\ \sin(t) \\ 0 \end{pmatrix}, \quad \operatorname{Im}(\mathbf{x}_1) = e^t \begin{pmatrix} -\sin(t) \\ 0 \\ \cos(t) \\ 0 \end{pmatrix}, \quad \operatorname{Re}(\mathbf{x}_2) = e^t \begin{pmatrix} t \cos(t) \\ \cos(t) \\ t \sin(t) \\ \sin(t) \end{pmatrix}, \quad \operatorname{Im}(\mathbf{x}_2) = e^t \begin{pmatrix} -t \sin(t) \\ -\sin(t) \\ t \cos(t) \\ \cos(t) \end{pmatrix}.$$

□

# Midterm #3

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

Good luck!

**Problem 1. (10 points)** Let  $A$  be a  $5 \times 5$  matrix with eigenvalues  $\pm 3i, 1, 1, 1$ .

- (a) Suppose that the eigenvalue  $\lambda = 1$  has defect 1. Does the equation  $\mathbf{x}' = A\mathbf{x}$  have (nonzero) solutions of one of the following forms?

$$(v_1 t + v_2) e^t \quad \left(v_1 \frac{t^2}{2} + v_2 t + v_3\right) e^t \quad \left(v_1 \frac{t^3}{6} + v_2 \frac{t^2}{2} + v_3 t + v_4\right) e^t \quad (v_1 t + v_2) \sin(3t) \quad v_1 e^t \cos(3t)$$

Circle those that are solutions (for appropriate choices of the coefficients  $v_1, v_2, v_3, v_4$ ).

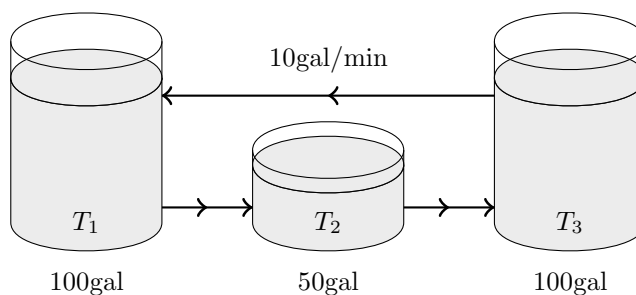
- (b) Now, consider the differential equation  $\mathbf{x}' = A\mathbf{x} + (3t^2, 0, \cos(t), 0, -1)^T$ . Write down a particular solution  $\mathbf{x}_p$  with undetermined coefficients.

**Problem 2. (10 points)** Three brine tanks  $T_1, T_2, T_3$  are connected as indicated in the sketch below.

The mixtures in each tank are kept uniform by stirring. Suppose that the mixture circulates between the tanks at the rate of 10gal/min.  $T_1$  and  $T_3$  contain 100gal of brine and  $T_2$  contains 50gal.

Denote by  $x_i(t)$  the amount (in pounds) of salt in tank  $T_i$  at time  $t$  (in minutes). Derive a system of linear differential equations for the  $x_i$ .

(Do *not* solve the system.)



**Problem 3. (20 points)** Let  $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ .

- (a) Find two linearly independent solutions to the linear system  $\mathbf{x}'(t) = A\mathbf{x}(t)$ .
- (b) Compute  $e^{tA}$ .

**Problem 4. (20 points)** Let  $A$  be a  $3 \times 3$  matrix such that  $e^{tA} = \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}$ .

(a) What are the eigenvalues of  $A$  and what are their defects?

(b) Solve the initial value problem  $\mathbf{x}'(t) = A\mathbf{x}(t)$ ,  $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

(c) Find a particular solution to the inhomogeneous linear system  $\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix}$ .

(d) Find the matrix  $A$ .

**Problem 5. (15 points)** Find four independent real-valued solutions of

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \mathbf{x}.$$

You may use that the characteristic polynomial has the repeated roots  $3 \pm 4i$ . Moreover, you may use that

$$\mathbf{v}_2 = (0 \ 0 \ 1 \ i)^T$$

is a generalized eigenvector of rank 2 for  $\lambda = 3 - 4i$ .

# Midterm #3

MATH 286 — Differential Equations Plus

Thursday, April 17

- No notes, personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **Explain your work!** Little or no points will be given for a correct answer with no explanation of how you got it.

**Good luck!**

**Problem 1. (10 points)** Let  $A$  be a  $5 \times 5$  matrix with eigenvalues  $\pm 3i, 1, 1, 1$ .

- (a) Suppose that the eigenvalue  $\lambda = 1$  has defect 1. Does the equation  $\mathbf{x}' = A\mathbf{x}$  have (nonzero) solutions of one of the following forms?

$$(v_1 t + v_2) e^t \quad \left(v_1 \frac{t^2}{2} + v_2 t + v_3\right) e^t \quad \left(v_1 \frac{t^3}{6} + v_2 \frac{t^2}{2} + v_3 t + v_4\right) e^t \quad (v_1 t + v_2) \sin(3t) \quad v_1 e^t \cos(3t)$$

Circle those that are solutions (for appropriate choices of the coefficients  $v_1, v_2, v_3, v_4$ ).

- (b) Now, consider the differential equation  $\mathbf{x}' = A\mathbf{x} + \begin{pmatrix} 3t^2 & 0 & \cos(t) & 0 & -1 \end{pmatrix}^T$ . Write down a particular solution  $\mathbf{x}_p$  with undetermined coefficients.

**Solution.**

- (a)  $(v_1 t + v_2) e^t$  is the only form, among the ones listed, of which there exists a nonzero solution.

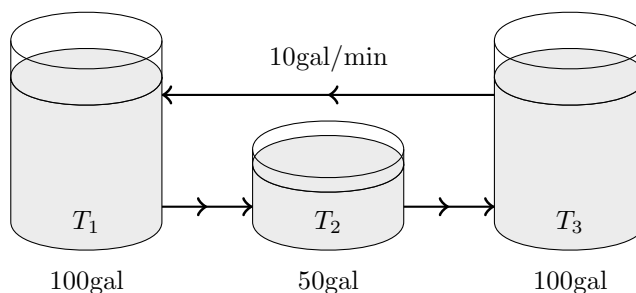
- (b)  $\mathbf{x}_p = \mathbf{a}_1 t^2 + \mathbf{a}_2 t + \mathbf{a}_3 + \mathbf{a}_4 \cos(t) + \mathbf{a}_5 \sin(t)$ . □

**Problem 2. (10 points)** Three brine tanks  $T_1, T_2, T_3$  are connected as indicated in the sketch below.

The mixtures in each tank are kept uniform by stirring. Suppose that the mixture circulates between the tanks at the rate of 10gal/min.  $T_1$  and  $T_3$  contain 100gal of brine and  $T_2$  contains 50gal.

Denote by  $x_i(t)$  the amount (in pounds) of salt in tank  $T_i$  at time  $t$  (in minutes). Derive a system of linear differential equations for the  $x_i$ .

(Do *not* solve the system.)



**Solution.** In the time interval  $[t, t + \Delta t]$ , we have:

$$\begin{aligned} \Delta x_1 &\approx 10 \cdot \frac{x_3}{100} \cdot \Delta t - 10 \cdot \frac{x_1}{100} \cdot \Delta t &\implies x'_1 &= \frac{1}{10} x_3 - \frac{1}{10} x_1 \\ \Delta x_2 &\approx 10 \cdot \frac{x_1}{100} \cdot \Delta t - 10 \cdot \frac{x_2}{50} \cdot \Delta t &\implies x'_2 &= \frac{1}{10} x_1 - \frac{1}{5} x_2 \\ \Delta x_3 &\approx 10 \cdot \frac{x_2}{50} \cdot \Delta t - 10 \cdot \frac{x_3}{100} \cdot \Delta t &\implies x'_3 &= \frac{1}{5} x_2 - \frac{1}{10} x_3 \end{aligned}$$

*Optional:* in matrix form, writing  $\mathbf{x} = (x_1, x_2, x_3)$ , this is

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{10} & 0 & \frac{1}{10} \\ \frac{1}{10} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{10} \end{pmatrix} \mathbf{x}.$$

□

**Problem 3. (20 points)** Let  $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ .

- (a) Find two linearly independent solutions to the linear system  $\mathbf{x}'(t) = A\mathbf{x}(t)$ .
- (b) Compute  $e^{tA}$ .

**Solution.**

- (a) The characteristic equation is  $(1 - \lambda)(-7 - \lambda) + 16 = \lambda^2 + 6\lambda + 9 = 0$ . So  $\lambda = -3$  is an eigenvalue of  $A$  with multiplicity 2. If  $\mathbf{v} = (a, b)^T$  is an eigenvector, then  $a = b$ , so the defect of  $\lambda$  is 1 and we must build a length 2 chain  $\{\mathbf{v}_1, \mathbf{v}_2\}$  of generalized eigenvectors. Taking  $\mathbf{v}_1 = (1, 1)^T$ , one then gets  $\mathbf{v}_2 = (A + 3I)\mathbf{v}_1$ , which has  $\mathbf{v}_2 = (1/4, 0)^T$  as a solution. We therefore obtain the following two linearly independent solutions:

$$\mathbf{x}_1(t) = e^{-3t} \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-3t} (t \mathbf{v}_1 + \mathbf{v}_2) = \begin{pmatrix} t + \frac{1}{4} \\ t \end{pmatrix} e^{-3t}.$$

- (b) From the first part, we get the fundamental matrix

$$\Phi(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t)) = e^{-3t} \begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^{tA} &= \Phi(t) \Phi(0)^{-1} = \begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix} e^{-3t} \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & 0 \end{pmatrix}^{-1} \\ &= e^{-3t} \begin{pmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix} \\ &= e^{-3t} \begin{pmatrix} 1 + 4t & -4t \\ 4t & 1 - 4t \end{pmatrix} = e^{-3t} \begin{pmatrix} 1 + 4t & -4t \\ 4t & 1 - 4t \end{pmatrix}. \end{aligned}$$

□



**Problem 4. (20 points)** Let  $A$  be a  $3 \times 3$  matrix such that  $e^{tA} = \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix}$ .

(a) What are the eigenvalues of  $A$  and what are their defects?

(b) Solve the initial value problem  $\mathbf{x}'(t) = A\mathbf{x}(t)$ ,  $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

(c) Find a particular solution to the inhomogeneous linear system  $\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix}$ .

(d) Find the matrix  $A$ .

**Solution.**

(a) After examining the columns of  $e^{tA}$  (which are linearly independent solutions to the given DE) and noting that any solution to the DE is a linear combination of eigenvalue solutions, we see that the only eigenvalue of  $A$  is  $\lambda = 0$  with multiplicity 3. The defect is 2 since terms of the form  $t e^{\lambda t}$ ,  $t^2 e^{\lambda t}$  appear in the columns of  $e^{tA}$ .

(b) The initial value problem is solved by

$$\mathbf{x}(t) = e^{tA} \mathbf{x}(0) = \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -t-t^2 \\ t-t^2 \\ 1 \end{pmatrix}.$$

(c) By variation of constants,

$$\begin{aligned} \mathbf{x}_p(t) &= e^{At} \int e^{-At} \begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix} dt = e^{At} \int \begin{pmatrix} 1-t & t & t-t^2 \\ -t & 1+t & -t+t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2/t^3 \\ 0 \end{pmatrix} dt \\ &= e^{At} \int \begin{pmatrix} 2/t^2 \\ 2/t^3 + 2/t^2 \\ 0 \end{pmatrix} dt \\ &= \begin{pmatrix} 1+t & -t & -t-t^2 \\ t & 1-t & t-t^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2/t \\ -1/t^2 - 2/t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2/t - 2 + 1/t + 2 \\ -2 - 1/t^2 - 2/t + 1/t + 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/t \\ -1/t^2 - 1/t \\ 0 \end{pmatrix} \end{aligned}$$

is a particular solution.

(d) We find  $A$  as

$$A = \left[ \frac{d}{dt} e^{tA} \right]_{t=0} = \left[ \begin{pmatrix} 1 & -1 & -1-2t \\ 1 & -1 & 1-2t \\ 0 & 0 & 0 \end{pmatrix} \right]_{t=0} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

**Problem 5. (15 points)** Find four independent real-valued solutions of

$$\mathbf{x}' = \begin{pmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{pmatrix} \mathbf{x}.$$

You may use that the characteristic polynomial has the repeated roots  $3 \pm 4i$ . Moreover, you may use that

$$\mathbf{v}_2 = (0 \ 0 \ 1 \ i)^T$$

is a generalized eigenvector of rank 2 for  $\lambda = 3 - 4i$ .

**Solution.** We first find the corresponding eigenvector  $\mathbf{v}_1$  as

$$\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_2 = \begin{pmatrix} 4i & -4 & 1 & 0 \\ 4 & 4i & 0 & 1 \\ 0 & 0 & 4i & -4 \\ 0 & 0 & 4 & 4i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}.$$

The chain induces the two solutions

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{v}_1 e^{(3-4i)t} = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} (\cos(4t) - i \sin(4t)) e^{3t} = \begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} + i \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t} \\ \mathbf{x}_2 &= (\mathbf{v}_1 t + \mathbf{v}_2) e^{(3-4i)t} = \begin{pmatrix} t \\ it \\ 1 \\ i \end{pmatrix} (\cos(4t) - i \sin(4t)) e^{3t} = \begin{pmatrix} \cos(4t) t \\ \sin(4t) t \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t} + i \begin{pmatrix} -\sin(4t) t \\ \cos(4t) t \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}. \end{aligned}$$

Taking real and imaginary part, this gives the four real-valued solutions:

$$\begin{pmatrix} \cos(4t) \\ \sin(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}, \quad \begin{pmatrix} -\sin(4t) \\ \cos(4t) \\ 0 \\ 0 \end{pmatrix} e^{3t}, \quad \begin{pmatrix} \cos(4t) t \\ \sin(4t) t \\ \cos(4t) \\ \sin(4t) \end{pmatrix} e^{3t}, \quad \begin{pmatrix} -\sin(4t) t \\ \cos(4t) t \\ -\sin(4t) \\ \cos(4t) \end{pmatrix} e^{3t}$$

□

Student Name: \_\_\_\_\_  
Student Net ID: \_\_\_\_\_

MATH 286 SECTION X1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 3

November 20, 2013

INSTRUCTOR: M. BRANNAN

**INSTRUCTIONS**

- This exam 60 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **EXPLAIN YOUR WORK!** Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- **GOOD LUCK!**

**PLEASE NOTE:** “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

**USEFUL FORMULAS:**

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \dots$$

$$\mathbf{x}(t) = \Phi(t)\Phi(a)^{-1}\mathbf{x}(a) + \Phi(t) \int_a^t \Phi(s)^{-1}\mathbf{f}(s)ds$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Question:	1	2	3	Total
Points:	12	14	24	50
Score:				

1. Consider the following first order linear system of differential equations:

$$x'_1 = -3x_1 + 2x_3$$

$$x'_2 = x_1 - x_2$$

$$x'_3 = -2x_1 - x_2.$$

- (a) (4 points) Write this system in the vector-matrix form  $\mathbf{x}' = A\mathbf{x}$ .

**Solution:**

$$\mathbf{x}' = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- (b) (8 points) The eigenvalues of the matrix  $A$  in part (a) are  $-2$  and  $-1 \pm (\sqrt{2})i$ . An eigenvector associated to the eigenvalue  $-1 - (\sqrt{2})i$  is

$$\mathbf{w} = \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1 - \sqrt{2}i \end{bmatrix}.$$

Find three linearly independent *real-valued* solutions to this system.

**Solution:** Let  $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  be an eigenvector for  $\lambda_1 = -2$ . Then

$$(A + 2I)\mathbf{v}_1 = 0 \iff \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

The first two equations imply that  $a = 2c = -b$ , while the third equation is the first minus the second. Taking  $a = 1$ , we get an eigenvector  $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1/2 \end{bmatrix}$ . This gives one solution  $\mathbf{x}_1(t) = e^{-2t}\mathbf{v}$ .

We are also given the eigenvector  $\mathbf{w}$  associated to the eigenvalue  $-1 - (\sqrt{2})i$ .

This yields a complex solution

$$\begin{aligned} \mathbf{z}(t) &= e^{(-1-\sqrt{2}i)t} \mathbf{w} = e^{-t} (\cos \sqrt{2}t - i \sin \sqrt{2}t) \begin{bmatrix} -\sqrt{2}i \\ 1 \\ -1 - \sqrt{2}i. \end{bmatrix} \\ &= e^{-t} \underbrace{\begin{bmatrix} -\sqrt{2} \sin \sqrt{2}t \\ \cos \sqrt{2}t \\ -\cos \sqrt{2}t - \sqrt{2} \sin \sqrt{2}t. \end{bmatrix}}_{\mathbf{x}_2(t)} + i e^{-t} \underbrace{\begin{bmatrix} -\sqrt{2} \cos \sqrt{2}t \\ -\sin \sqrt{2}t \\ -\sqrt{2} \cos \sqrt{2}t + \sin \sqrt{2}t. \end{bmatrix}}_{\mathbf{x}_3(t)} \end{aligned}$$

Then  $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$  are three linearly independent solutions real valued solutions.

2. (a) (10 points) Let  $\lambda$  be a fixed real number, and let

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Show that  $e^{tA} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$

**Solution:** Write  $tA = \lambda tI + tN$ , where

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies N^3 = 0.$$

From this, we get

$$e^{tA} = e^{\lambda tI + tN} = e^{\lambda tI} e^{tN} = e^{\lambda t} I e^{tN} = e^{\lambda t} (I + tN + \frac{t^2}{2} N^2) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}.$$

An alternate solution to this problem would be to show that the matrix  $\Phi(t) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$  is a fundamental matrix for the system  $\mathbf{x}' = A\mathbf{x}$ , and that  $\Phi(0) = I$ . Then  $e^{tA} = \Phi(t)\Phi(0)^{-1} = \Phi(t)$ .

- (b) (4 points) Let  $A$  be the matrix from part (a). Solve the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t); \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

**Solution:** The solution is

$$\mathbf{x}(t) = e^{tA} \mathbf{x}(0) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 + 2t + \frac{3t^2}{2} \\ 2 + 3t \\ 3 \end{bmatrix}.$$

3. (a) (9 points) Find two linearly independent solutions to the system

$$\mathbf{x}'(t) = A\mathbf{x}(t); \quad \text{where} \quad A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$$

**Solution:** First, we find eigenvalues:

$$0 = \det(A - \lambda I) = (7 - \lambda)(3 - \lambda) + 4 = 21 - 10\lambda + \lambda^2 + 4 = \lambda^2 - 10\lambda + 25.$$

So  $\lambda = 5$  is a multiplicity 2 eigenvalue. Next, we look for eigenvectors: Let  $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$  be an eigenvector, then

$$(A - 5I)\mathbf{v} = 0 \iff \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \mathbf{v} = \begin{bmatrix} a \\ -2a \end{bmatrix} \quad (a \neq 0).$$

From this we see that  $\lambda = 5$  has defect 1 and we want to find a length 2 chain  $\{\mathbf{v}_1, \mathbf{w}_2\}$  of generalized eigenvectors. Taking  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ , we then have for  $\mathbf{v}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ ,

$$\mathbf{v}_1 = (A - 5I)\mathbf{v}_2 \iff \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \iff \begin{aligned} 1 &= 2a + b \\ -2 &= -4a - 2b \end{aligned}$$

Taking  $a = 1/2$  and  $b = 0$ , we get  $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ . This gives the following two linearly independent solutions:

$$\mathbf{x}_1(t) = e^{5t}\mathbf{v}_1 \quad \& \quad \mathbf{x}_2(t) = e^{5t}(t\mathbf{v}_1 + \mathbf{v}_2).$$

- (b) (3 points) Write down a fundamental matrix  $\Phi(t)$  for the system in part (a).

**Solution:** Let  $\Phi(t) = [\mathbf{x}_1(t) \ \mathbf{x}_2(t)] = \begin{bmatrix} e^{5t} & e^{5t}(t + 1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix}.$

- (c) (5 points) Compute the matrix exponential  $e^{tA}$ , where  $A$  is the matrix from part (a).

**Solution:**

$$\begin{aligned} e^{tA} &= \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{5t} & e^{5t}(t+1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ -2 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} e^{5t} & e^{5t}(t+1/2) \\ -2e^{5t} & e^{5t}(-2t) \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \end{aligned}$$

- (d) (7 points) Solve the following initial value problem:

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}; \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where  $A$  is the matrix from part (a).

**Solution:** We will use the matrix exponential from part (c).

$$\begin{aligned} \mathbf{x}(t) &= e^{tA}\mathbf{x}(0) + e^{tA} \int_0^t e^{-sA} \begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds \\ &= e^{tA}(\mathbf{x}(0) + \int_0^t e^{-sA} \begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-5s}(-2s+1) & -se^{-5s} \\ e^{-5s}(4s) & e^{-5s}(2s+1) \end{bmatrix} \begin{bmatrix} e^{-s} \\ 0 \end{bmatrix} ds \right) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-6s}(-2s+1) \\ e^{-6s}(-4s) \end{bmatrix} ds \right) \\ &= \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{9}e^{-6t}(3t-1) + \frac{1}{9} \\ \frac{-1}{9}e^{-6t}(6t+1) + \frac{1}{9} \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{5t}(3t+1) \\ e^{5t}(-6t+1) \end{bmatrix} + \begin{bmatrix} e^{5t}(2t+1) & te^{5t} \\ e^{5t}(-4t) & e^{5t}(-2t+1) \end{bmatrix} \begin{bmatrix} \frac{1}{9}e^{-6t}(3t-1) + \frac{1}{9} \\ \frac{-1}{9}e^{-6t}(6t+1) + \frac{1}{9} \end{bmatrix} \\ &= \begin{bmatrix} \frac{10}{9}e^{5t}(3t+1) \\ \frac{10}{9}e^{5t}(-6t+1) \end{bmatrix} + \frac{e^{-t}}{9} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \end{aligned}$$



**(BONUS PROBLEM (5 Points)).**

Let  $\lambda$  be an eigenvalue of an  $n \times n$  matrix  $A$  and let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  be a length  $r$  chain of generalized eigenvectors associated to the eigenvalue  $\lambda$ .

(a). Explain what it means to be a length  $r$  chain of generalized eigenvectors.

*For  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  to be a length  $r$  chain of generalized eigenvectors, the vector  $\mathbf{v}_r$  must satisfy  $(A - \lambda I)^r \mathbf{v}_r = 0$ ,  $(A - \lambda I)^{r-1} \mathbf{v}_r \neq 0$ , and the remaining vectors  $\mathbf{v}_1, \dots, \mathbf{v}_{r-1}$  are then given by*

$$\mathbf{v}_s = (A - \lambda I)^{r-s} \mathbf{v}_r \neq 0 \quad (1 \leq s \leq r-1).$$

(b). Show that the vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$  are linearly independent. (**Hint:** Suppose that  $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_r \mathbf{v}_r = 0$ . Multiply this equation by  $(A - \lambda I)$ ,  $(A - \lambda I)^2$ ,  $(A - \lambda I)^3$ , etc... and see what happens.)

*Suppose that  $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_r \mathbf{v}_r = 0$ . Note that*

$$(A - \lambda I)^k \mathbf{v}_s = 0 \quad (k \geq s),$$

*and*

$$(A - \lambda I)^k \mathbf{v}_s = \mathbf{v}_{s-k} \quad (s > k).$$

*Therefore if we take the above equation and multiply it by  $(A - \lambda I)^k$  for  $k = 1, 2, \dots, r-1$ , we get the following system of equations*

$$\begin{aligned} 0 &= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_r \mathbf{v}_r \\ 0 &= c_2 \mathbf{v}_1 + \dots + c_r \mathbf{v}_{r-1} \quad (\text{mult. by } A - \lambda I) \\ 0 &= c_3 \mathbf{v}_1 + \dots + c_r \mathbf{v}_{r-2} \quad (\text{mult. by } (A - \lambda I)^2) \\ &\dots \\ 0 &= c_{r-1} \mathbf{v}_1 + c_r \mathbf{v}_2 \quad (\text{mult. by } (A - \lambda I)^{r-2}) \\ \implies 0 &= c_r \mathbf{v}_1 \quad (\text{mult. by } (A - \lambda I)^{r-1}) \end{aligned}$$

*The last equation implies that  $c_r = 0$ , the second last then implies that  $c_{r-1} = 0$ , and continuing up the list of equations, we see that  $c_1 = c_2 = \dots = c_r = 0$ . Therefore the given vectors are linearly independent.*

*(Extra work space.)*

*Read Euler, read Euler, he is the master of us all.*  
— Pierre-Simon Laplace (1749–1827) —

**Problem 1.**

- (a) Find the Fourier series of the function of period 2 characterized by

$$f(t) = \begin{cases} t, & \text{for } 0 \leq t < 1, \\ t+2, & \text{for } 1 \leq t < 2. \end{cases}$$

- (b) Let  $g(t)$  be the sum of the Fourier series you just calculated. Sketch the graph of  $g(t)$ . What are  $g(0)$ ,  $g(1)$  and  $g(2)$ ? Explain the general phenomenon.

**Problem 2.** A mass-spring system is described by the equation

$$mx'' + x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

- (a) For which  $m$  does pure resonance occur?  
(b) Find the general solution when  $m = 1/9$ .

**Problem 3.** Let  $f(t) = 1$  for  $t \in (0, L)$ .

- (a) Extend  $f(t)$  to an odd  $2L$ -periodic function  $f_o(t)$ . Sketch the graph of the sum of the Fourier series of  $f_o(t)$ .  
(b) Calculate the Fourier series of  $f_o(t)$  with period  $2L$ . (This is also known as the Fourier sine series of  $f(t)$ .)  
(c) Explain, using the heat equation as an example, why it can be useful to write a *constant function* as an infinite sum of sine terms.

**Problem 4.** For which values of  $\lambda$  does the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

have nonzero solutions? Find all these solutions. Make sure to consider all cases.

**Problem 5.** Find the solution  $u(x, t)$ , for  $0 < x < 3$  and  $t \geq 0$ , to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0, t) = 0, \quad u(3, t) = 0, \quad u(x, 0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to the previous problem). Don't rely on a formula.

**Problem 6.** Using the Laplace transform, solve the initial value problem  $x'' + 4x' + 4x = f(t)$  with  $x(0) = 0$ ,  $x'(0) = 0$  and

$$f(t) = \begin{cases} 2, & \text{for } 0 \leq t < 2, \\ t, & \text{for } 2 \leq t < 3, \\ 1, & \text{for } t \geq 3. \end{cases}$$

Finally, here is the table for the Laplace transform, which you will be given for the final exam.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{at}$	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-sa}F(s)$

*Read Euler, read Euler, he is the master of us all.*  
— Pierre-Simon Laplace (1749–1827) —

## Problem 1.

- (a) Find the Fourier series of the function of period 2 characterized by

$$f(t) = \begin{cases} t, & \text{for } 0 \leq t < 1, \\ t+2, & \text{for } 1 \leq t < 2. \end{cases}$$

- (b) Let  $g(t)$  be the sum of the Fourier series you just calculated. Sketch the graph of  $g(t)$ . What are  $g(0)$ ,  $g(1)$  and  $g(2)$ ? Explain the general phenomenon.

## Solution.

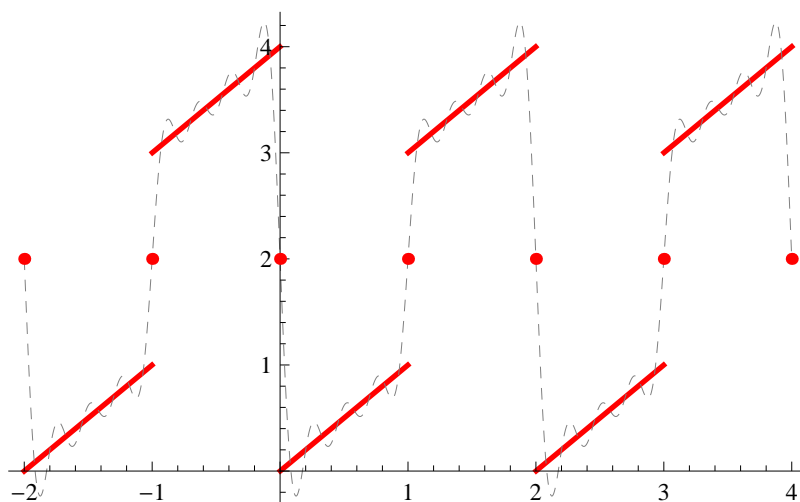
- (a) We compute:

$$\begin{aligned} a_0 &= \int_0^2 f(t) dt = \int_0^1 t dt + \int_1^2 (t+2) dt = 4 \\ a_m &= \int_0^2 f(t) \cos(m\pi t) dt = \int_0^1 t \cos(m\pi t) dt + \int_1^2 2 \cos(m\pi t) dt = \dots = 0 \\ b_m &= \int_0^2 f(t) \sin(m\pi t) dt = \int_0^1 t \sin(m\pi t) dt + \int_1^2 2 \sin(m\pi t) dt = \dots = \frac{2((-1)^m - 2)}{m\pi} \end{aligned}$$

Note that  $a_m = 0$  for  $m > 0$  follows (and we could have avoided the calculation by noticing) from the fact that  $f(t) - \frac{a_0}{2} = f(t) - 2$  is an odd function (make a plot to convince yourself!). The Fourier series is:

$$2 + \sum_{m=1}^{\infty} \frac{2((-1)^m - 2)}{m\pi} \sin(m\pi t)$$

- (b) Graph of  $g(t)$  in red (with an approximation in light dashed gray):



We have  $g(0) = g(1) = g(2) = 2$ . These values are the averages between the limit from the left and the limit from the right (you can see from the dashed approximation of the Fourier series why this general behaviour of Fourier series makes sense).  $\square$

**Problem 2.** A mass-spring system is described by the equation

$$mx'' + x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

- (a) For which  $m$  does pure resonance occur?
- (b) Find the general solution when  $m = 1/9$ .

**Solution.**

- (a) Characteristic equation:  $mr^2 + 1 = 0$  with roots  $r = \pm i\sqrt{1/m}$ .

Natural frequency is  $\sqrt{1/m}$ .

The external frequencies are  $n/3$  where  $n$  is an odd positive integer.

$$\sqrt{1/m} = n/3 \iff m = 9/n^2 \text{ (since } m > 0\text{)}$$

Pure resonance occurs if  $m = 9/n^2$  for an odd integer  $n \geq 1$  (that is,  $m = 9, 1, 9/25, 9/49, \dots$ ).

- (b) In this case, the natural frequency is 3 and we have pure resonance because  $3 = n/3$  for  $n = 9$ . For  $n \neq 9$  we solve

$$\frac{1}{9}x'' + x = \frac{1}{n^2} \sin\left(\frac{nt}{3}\right).$$

This has a solution of the form  $x_p = A \cos\left(\frac{nt}{3}\right) + B \sin\left(\frac{nt}{3}\right)$  where  $A, B$  are undetermined. Plugging into the DE:

$$\frac{1}{9}x_p'' + x_p = A\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\cos\left(\frac{nt}{3}\right) + B\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)\sin\left(\frac{nt}{3}\right) \stackrel{!}{=} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right)$$

It follows that  $A = 0$  (we could have seen that coming...) and

$$B = \frac{1}{n^2\left(-\frac{1}{9}\frac{n^2}{9} + 1\right)} = \frac{81}{n^2(81 - n^2)}, \quad x_p = \frac{81}{n^2(81 - n^2)} \sin\left(\frac{nt}{3}\right).$$

The case  $n = 9$  has to be done separately: because of resonance there now exists a solution of the form

$$x_p = At \cos(3t) + Bt \sin(3t).$$

Plugging into the DE:

$$\frac{1}{9}x_p'' + x_p = \frac{2}{3}B \cos(3t) - \frac{2}{3}A \sin(3t) \stackrel{!}{=} \frac{1}{81} \sin(3t)$$

It follows that  $B = 0$  and  $A = -\frac{1}{54}$ . So  $x_p = -\frac{1}{54}t \cos(3t)$ . By superposition it follows that

$$\frac{1}{9}x'' + x = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} \sin\left(\frac{nt}{3}\right) \quad \text{has solution} \quad x_p = -\frac{1}{54}t \cos(3t) + \sum_{\substack{n=1 \\ n \text{ odd}, n \neq 9}} \frac{81}{n^2(81 - n^2)} \sin\left(\frac{nt}{3}\right).$$

The general solution is  $x(t) = x_p(t) + A \cos(3t) + B \sin(3t)$ . □

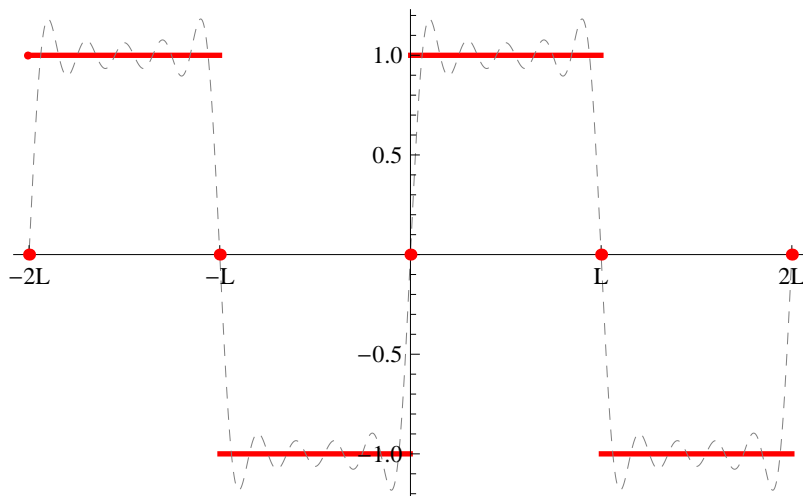
**Problem 3.** Let  $f(t) = 1$  for  $t \in (0, L)$ .

- (a) Extend  $f(t)$  to an odd  $2L$ -periodic function  $f_o(t)$ . Sketch the graph of the sum of the Fourier series of  $f_o(t)$ .
- (b) Calculate the Fourier series of  $f_o(t)$  with period  $2L$ . (This is also known as the Fourier sine series of  $f(t)$ .)

- (c) Explain, using the heat equation as an example, why it can be useful to write a *constant function* as an infinite sum of sine terms.

**Solution.**

- (a) Sketch of the Fourier series in red (with an approximation in light dashed gray):



- (b) The odd  $2L$ -periodic extension of  $f(t)$  takes the values  $f(t) = \begin{cases} -1 & \text{for } t \in (-L, 0) \\ +1 & \text{for } t \in (0, L) \end{cases}$ .

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi t}{L}\right) dt = \frac{2}{L} \left[ -\frac{L}{\pi n} \cos\left(\frac{n\pi t}{L}\right) \right]_0^L = \frac{2}{\pi n} [1 - \cos(n\pi)] \\ &= \frac{2}{\pi n} [1 - (-1)^n] = \begin{cases} \frac{4}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Thus the Fourier series is:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin\left(\frac{n\pi t}{L}\right)$$

- (c) When solving the heat equation we found that the equations

$$\begin{aligned} u_t &= k u_{xx} & t > 0, \quad x \in (0, L) \\ u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned}$$

have the solutions  $u_n(x, t) = e^{-\frac{n^2 \pi^2}{L^2} k t} \sin\left(\frac{n\pi}{L} x\right)$ . The superposition  $u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t)$  then satisfies

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L} x\right).$$

This is the initial temperature distribution and the  $c_n$  are its Fourier (sine) coefficients! In applications,  $u(x, 0)$  is often given to us; for instance,  $u(x, 0) = 1$ . In this case, we need to know the Fourier sine expansion of 1 on  $(0, L)$  in order to find the coefficients  $c_n$ .  $\square$

**Problem 4.** For which values of  $\lambda$  does the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$$

have nonzero solutions? Find all these solutions. Make sure to consider all cases.

**Solution.** To solve this EVP, we distinguish three cases:

$\lambda < 0$ . Then the roots are the real numbers  $\pm r = \pm\sqrt{-\lambda}$  and the general solution to the DE is  $y(x) = Ae^{rx} + Be^{-rx}$ . Then  $y'(0) = Ar - Br = 0$  implies  $B = A$ , so that  $y(3) = A(e^{3r} + e^{-3r})$ . Since  $e^{3r} + e^{-3r} > 0$ , we see that  $y(3) = 0$  only if  $A = 0$ . So there is no solution to the EVP for  $\lambda < 0$ .

$\lambda = 0$ . Now the general solution to the DE is  $y(x) = A + Bx$ . Then  $y'(0) = 0$  implies  $B = 0$ , and it follows from  $y(3) = A = 0$  that  $\lambda = 0$  is not an eigenvalue.

$\lambda > 0$ . Now the roots are  $\pm i\sqrt{\lambda}$  and  $y(x) = A \cos(\sqrt{\lambda}x) + B \sin(\sqrt{\lambda}x)$ .  $y'(0) = B\sqrt{\lambda} = 0$  implies  $B = 0$ . Then  $y(3) = A \cos(3\sqrt{\lambda}) = 0$ . Note that  $\cos(3\sqrt{\lambda}) = 0$  is true if and only if  $3\sqrt{\lambda} = \frac{(2n+1)\pi}{2}$  for some integer  $n$ . In that case,  $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$  and  $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ .

In summary, this means that the only nonzero solutions are the functions  $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ , corresponding to  $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$ , with  $n = 0, 1, 2, \dots$  (why did we include  $n = 0$  but excluded  $n = -1, -2, \dots$ ?).  $\square$

**Problem 5.** Find the solution  $u(x, t)$ , for  $0 < x < 3$  and  $t \geq 0$ , to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0, t) = 0, \quad u(3, t) = 0, \quad u(x, 0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to the previous problem). Don't rely on a formula.

**Solution.** We look for solutions to (that's the homogeneous/linear parts of the problem at hand)

$$2u_t = u_{xx}, \quad u_x(0, t) = u(3, t) = 0$$

which are of the form  $u(x, t) = X(x)T(t)$ . The boundary conditions imply  $X'(0) = 0$  and  $X(3) = 0$ .

$$2X(x)T'(t) = X''(x)T(t) \implies \frac{X''(x)}{X(x)} = \frac{2T'(t)}{T(t)} = \text{const} =: -\lambda$$

In particular,  $X(x)$  is a solution to the eigenvalue problem

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X(3) = 0.$$

By the previous problem,  $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$  and  $X(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ , with  $n = 0, 1, 2, \dots$

$T$  solves  $2T' + \lambda T = 0$ , and hence, up to multiples,  $T(t) = e^{-\frac{1}{2}\lambda t} = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}$ .

Taken together, we have the solutions  $u_n(x, t) = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t} \cos\left(\frac{(2n+1)\pi}{6}x\right)$  solving  $2u_t = u_{xx}$  and  $u_x(0, t) = u(3, t) = 0$ .

Note that  $u_n(x, 0) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ . In particular, the superposition

$$u(x, t) = 2u_1(x, t) + 7u_4(x, t) = 2e^{-\frac{1}{8}\pi^2 t} \cos\left(\frac{\pi x}{2}\right) + 7e^{-\frac{9}{8}\pi^2 t} \cos\left(\frac{3\pi x}{2}\right)$$

solves our heat conduction problem.

[It is not obvious that *every* initial temperature distribution  $f(x)$  can be written as a (infinite) superposition of the  $u_n(x, 0)$ . However, such "eigenfunction expansions" are always possible (thus extending what we know about ordinary Fourier series). See Chapter 10 in case you are interested to learn more.]  $\square$



**Problem 6.** Using the Laplace transform, solve the initial value problem  $x'' + 4x' + 4x = f(t)$  with  $x(0) = 0$ ,  $x'(0) = 0$  and

$$f(t) = \begin{cases} 2, & \text{for } 0 \leq t < 2, \\ t, & \text{for } 2 \leq t < 3, \\ 1, & \text{for } t \geq 3. \end{cases}$$

**Solution.** Let  $X(s)$  be the Laplace transform of  $x(t)$ . Note that

$$f(t) = 2(u_0(t) - u_2(t)) + t(u_2(t) - u_3(t)) + u_3(t) = 2 + u_2(t)(t - 2) - u_3(t)((t - 3) + 2).$$

An application of the Laplace transform yields

$$s^2X + 4sX + 4X = \frac{2}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - 2\frac{e^{-3s}}{s}.$$

It follows that

$$X(s) = 2(1 - e^{-3s})\frac{1}{s(s+2)^2} + (e^{-2s} - e^{-3s})\frac{1}{s^2(s+2)^2}.$$

Using partial fractions, we have

$$\begin{aligned} X(s) &= (1 - e^{-3s})\frac{1}{2}\left(\frac{1}{s} - \frac{2}{(s+2)^2} - \frac{1}{s+2}\right) + (e^{-2s} - e^{-3s})\frac{1}{4}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+2)^2} + \frac{1}{s+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{2}{(s+2)^2} - \frac{1}{s+2}\right) + \frac{e^{-2s}}{4}\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+2)^2} + \frac{1}{s+2}\right) - \frac{e^{-3s}}{4}\left(\frac{1}{s^2} + \frac{1}{s} - \frac{3}{(s+2)^2} - \frac{1}{s+2}\right). \end{aligned}$$

Taking the inverse transform, we obtain

$$\begin{aligned} x(t) &= \frac{1}{2}(1 - 2te^{-2t} - e^{-2t}) + \frac{u_2(t)}{4}((t-2) - 1 + (t-2)e^{-2(t-2)} + e^{-2(t-2)}) \\ &\quad - \frac{u_3(t)}{4}((t-3) + 1 - 3(t-3)e^{-2(t-3)} - e^{-2(t-3)}) \\ &= \frac{1}{2}(1 - (2t+1)e^{-2t}) + \frac{u_2(t)}{4}(t-3 + (t-1)e^{-2(t-2)}) - \frac{u_3(t)}{4}(t-2 - (3t-8)e^{-2(t-3)}). \end{aligned}$$

□

Finally, here is the table for the Laplace transform, which you will be given for the final exam.

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{at}$	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$t^n$	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-sa}F(s)$

Student Name: \_\_\_\_\_  
Student Net ID: \_\_\_\_\_

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN  
DEPARTMENT OF MATHEMATICS  
MATH 286 SECTION X1 – Introduction to Differential Equations Plus  
FINAL EXAMINATION  
DECEMBER 17, 2013  
INSTRUCTOR: M. BRANNAN

**INSTRUCTIONS**

- This exam is three (3) hours long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- **EXPLAIN YOUR WORK!** Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- **GOOD LUCK!**

**PLEASE NOTE:** “Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.”

Question:	1	2	3	4	5	6	7	Total
Points:	14	25	9	9	14	15	14	100
Score:								

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**SOME USEFUL FORMULAS:**

$$e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k = I + B + \frac{1}{2!} B^2 + \frac{1}{3!} B^3 + \dots$$

$$\mathbf{x}(t) = \Phi(t)\Phi(a)^{-1}\mathbf{x}(a) + \Phi(t) \int_a^t \Phi(s)^{-1}\mathbf{f}(s)ds$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt$$

---

1. Let  $f(t)$  be the  $\pi$ -periodic function defined by

$$f(t) = \begin{cases} 1 & -\frac{\pi}{2} < t < 0, \\ t & 0 \leq t \leq \frac{\pi}{2}. \end{cases}$$

- (a) (2 points) Sketch the graph of  $f$  over a few periods.

- (b) (6 points) Let  $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L})$  be the Fourier series for  $f$ . Calculate the Fourier coefficients  $a_0$ ,  $a_n$  and  $b_n$  ( $n \geq 1$ ).

**Solution:**

$$a_0 = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(t) dt = \frac{2}{\pi} \left( \frac{\pi}{2} + \frac{\pi^2}{8} \right) = 1 + \frac{\pi}{4},$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \cos(2nt) dt = \frac{2}{\pi} \int_{-\pi/2}^0 \cos(2nt) dt + \frac{2}{\pi} \int_0^{\pi/2} t \cos(2nt) dt \\ &= \frac{\sin(2nt)}{n\pi} \Big|_{-\pi/2}^0 + \frac{2}{\pi} \left( \frac{t \sin(2nt)}{2n} + \frac{\cos(2nt)}{4n^2} \right) \Big|_0^{\pi/2} \\ &= \frac{\cos(n\pi) - 1}{2\pi n^2}, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} f(t) \sin(2nt) dt = \frac{2}{\pi} \int_{-\pi/2}^0 \sin(2nt) dt + \frac{2}{\pi} \int_0^{\pi/2} t \sin(2nt) dt \\ &= \frac{-\cos(2nt)}{n\pi} \Big|_{-\pi/2}^0 + \frac{2}{\pi} \left( \frac{-t \cos(2nt)}{2n} + \frac{\sin(2nt)}{4n^2} \right) \Big|_0^{\pi/2} \\ &= \frac{\cos(n\pi) - 1}{n\pi} - \frac{\cos(n\pi)}{2n}. \end{aligned}$$

- (c) (2 points) Does the Fourier series for  $f$  converge to  $f(t)$  at every point  $t$ ? What does the Fourier series converge to when  $t = 0$ ?

**Solution:** Since  $f$  is piecewise smooth and has jump discontinuities, the FS for  $f$  does not converge at every point  $t$ . For example, at  $t = 0$  the FS converges to  $\frac{f(0^+) + f(0^-)}{2} = 1/2 \neq f(0)$ .

- (d) (4 points) A 1 kg cart is connected to a wall by a spring with unknown spring constant  $k > 0$  N/m, and is periodically forced by  $f(t)$  Newtons (where  $f$  is the periodic function defined above). Assuming there is no friction in the system, the resulting equation of motion for the displacement  $x(t)$  of the cart from rest is given by

$$x'' + kx = f(t).$$

Find all values of  $k$  that will cause resonance in the forced mechanical system.

**Solution:** When the Fourier Series  $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2nt) + b_n \sin(2nt))$  for  $f$  contains a non-zero sine or cosine component with frequency equal to the natural frequency  $\omega_0 = \sqrt{k}$  of the system, there will be resonance. Since all the  $b_n$ 's (and all the odd  $a_n$ 's) are non-zero, this happens precisely when

$$\sqrt{k} = 2n \iff k = 4n^2 \quad (n = 1, 2, 3, \dots).$$

2. In this multi-part problem, we will derive the solution to a one-dimensional heat equation with *mixed boundary conditions* (with one endpoint held at a fixed temperature, and the other endpoint insulated). For the remainder of this problem, let  $L > 0$  be fixed.

(a) (6 points) Consider the constant function

$$f(x) = 100 \quad \text{defined on the interval } [0, 2L].$$

Sketch the graph of the  $4L$ -**periodic odd extension** of  $f$  and compute its Fourier sine series.

**Solution:** The expansion we seek is  $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2L}$ , where

$$\begin{aligned} b_n &= \frac{2}{2L} \int_0^{2L} f(x) \sin \frac{n\pi x}{2L} dx = \frac{100}{L} \int_0^{2L} \sin \frac{n\pi x}{2L} dx \\ &= \frac{-200}{n\pi} \cos \frac{n\pi x}{2L} \Big|_0^{2L} = \begin{cases} 0 & n \text{ is even} \\ \frac{400}{n\pi} & n \text{ is odd} \end{cases} \end{aligned}$$

- (b) (5 points) Consider the following eigenvalue problem for the function  $X(x)$  on the interval  $[0, L]$ :

$$X'' + \lambda X = 0; \quad X(0) = X'(L) = 0.$$

Show that the  $\lambda$  is an eigenvalue if and only if  $\lambda = \lambda_n = \frac{(2n-1)^2 \pi^2}{4L^2}$ , where  $n = 1, 2, 3, \dots$ . For each  $\lambda_n$ , write down the corresponding eigenfunction  $X_n(x)$ .

(**Note:** You may assume without proof that all the eigenvalues are *positive*.)

**Solution:** Since  $\lambda > 0$ , the general solution to this ODE is

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

From the BC's, we get

$$c_1 = 0, \quad \& \quad \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}2L) = 0 \implies \sqrt{\lambda}2L = (2n-1)\pi \iff \lambda = \frac{(2n-1)^2 \pi^2}{4L^2},$$

where  $n = 1, 2, \dots$ . The corresponding eigenfunction is  $X_n(x) = \sin \frac{(2n-1)\pi x}{2L}$ .

- (c) (3 points) A laterally insulated metal rod of length  $L$  (with thermal diffusivity  $k = 2$ ) is heated to a uniform temperature of 100 degrees Celsius. At time  $t = 0$ , the left end of the rod ( $x = 0$ ) is placed in an ice bath at 0 degrees Celsius, and the right end ( $x = L$ ) is insulated so that no heat flows in or out at this end. If  $u(x, t)$  denotes the temperature (in degrees Celsius) of the rod at position  $0 < x < L$  and time  $t > 0$ , then  $u$  satisfies the one-dimensional heat equation

$$u_t = 2u_{xx} \quad (0 < x < L, \ t > 0).$$

Write down the Boundary Conditions and Initial Condition for this problem.

**Solution:**

$$u(0, t) = 0 = u_x(L, t) = 0 \quad \& \quad u(x, 0) = 100.$$

- (d) (4 points) Using the method of **separation of variables**, show that if

$$u(x, t) = X(x)T(t)$$

is a solution to the above heat equation satisfying the boundary conditions from part (c), then  $X(x)$  must be a solution to the eigenvalue problem in part (b).

**Solution:** If  $u(x, t) = X(x)T(t)$ , then

$$u_t = 2u_{xx} \iff XT' = 2X''T \iff \frac{X''}{X} = \frac{T'}{2T} = -\lambda \quad (\lambda \in \mathbb{R}).$$

This together with the BCs above yields the two ODEs

$$X'' + \lambda X = 0; \quad X(0) = X'(L) = 0$$

and

$$T' + 2\lambda T = 0.$$

- (e) (3 points) For each eigenfunction  $X_n(x)$  from part (b), find the corresponding solution  $T_n(t)$ .

**Solution:**

$$T'_n + 2\frac{(2n-1)^2\pi^2}{4L^2}T_n = 0 \implies T_n(t) = \exp\left(-2\frac{(2n-1)^2\pi^2 t}{4L^2}\right)$$

(f) (4 points) Let

$$u(x, t) = \sum_{n=1}^{\infty} \alpha_n X_n(x) T_n(t).$$

Then  $u(x, t)$  satisfies the heat equation and boundary conditions from part (c). Find the constants  $\alpha_n$  so that  $u(x, t)$  also satisfies the initial condition  $u(x, 0)$ . (**Hint:** Use part (a).)

**Solution:** We have

$$u(x, t) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L} \exp \left( -2 \frac{(2n-1)^2 \pi^2 t}{4L^2} \right),$$

so

$$100 = u(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}.$$

This tells us that

$$\alpha_n = b_{2n-1} \quad (n = 1, 2, \dots).$$



3. (9 points) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x + 3y}{x - y}; \quad y(1) = 0.$$

An implicit equation for  $y(x)$  is fine.

**Solution:** Let  $v = \frac{y}{x}$ . Then  $\frac{dy}{dx} = xv' + v$  and the equation becomes

$$xv' + v = \frac{1 + 3v}{1 - v} \iff xv' = \frac{1 + 2v + v^2}{1 - v} \iff \frac{(1 - v)dv}{(v + 1)^2} = \frac{dx}{x}.$$

Integrating, this gives

$$\frac{-2}{v + 1} - \ln |v + 1| = \ln |x| + C \iff \frac{-2}{y/x + 1} - \ln |y/x + 1| = \ln |x| + C.$$

Since  $y(1) = 0$ , we get  $-2 - 0 = 0 + C$ , so the solution is

$$\frac{-2}{y/x + 1} - \ln |y/x + 1| = \ln |x| - 2.$$

4. Consider the ODE

$$y + (2x - e^y) \frac{dy}{dx} = 0.$$

(a) (3 points) Is this equation exact?

**Solution:** Let  $M = y$  and  $N = 2x - e^y$ . Clearly  $M_y = 1 \neq 2 = N_x$ , so the equation is not exact.

(b) (6 points) Find an implicit expression for the general solution to this ODE.  
(**HINT:** Multiply the above ODE by  $y$  and then check for exactness).

**Solution:** Multiplying by  $y$ , the ODE becomes

$$y^2 + (2xy - ye^y) \frac{dy}{dx} = 0.$$

Let  $M = y^2$  and  $N = 2xy - ye^y$ . Then  $M_y = 2y = N_x$ , so the equation is exact. To solve the ODE, we find an  $F(x, y)$  such that  $F_x = M$  and  $F_y = N$ . Then  $F(x, y) = C$  will be an implicit solution. Now,

$$F(x, y) = \int M dx = \int y^2 dx = xy^2 + g(y).$$

To find  $g(y)$ , note that

$$2xy + g'(y) = F_y = N = 2xy - ye^y \implies g'(y) = -ye^y \implies g(y) = -ye^y + e^y.$$

Thus,  $xy^2 - ye^y + e^y = C$  is a solution.

5. (a) (3 points) Find the general solution to the ODE

$$y'' - 10y' + 21y = 0.$$

**Solution:** Let  $P(r) = r^2 - 10r + 21 = (r - 7)(r - 3)$  be the characteristic polynomial. Then the general solution is

$$y(x) = c_1 e^{7x} + c_2 e^{3x}.$$

- (b) (6 points) Solve the initial value problem

$$y'' - 10y' + 21y = e^{3x} + e^x; \quad y(0) = y'(0) = 0.$$

**Solution:** Let  $y_p(x) = A x e^{3x} + B e^x$  be a trial particular solution. Plugging this in, we get

$$6A e^{3x} + 9A x e^{3x} + B e^x - 10(A e^{3x} + 3A x e^{3x} + B e^x) + 21(A x e^{3x} + B e^x) = e^{3x} + e^x.$$

This gives

$$6A - 10A = 1 \implies A = -\frac{1}{4} \quad \& \quad B - 10B + 21B = 1 \implies B = \frac{1}{12}.$$

The general solution to this ODE is then

$$y(x) = c_1 e^{7x} + c_2 e^{3x} - \frac{1}{4} x e^{3x} + \frac{1}{12} e^x.$$

Plugging in the IC's we get  $c_1 = \frac{5}{48}$ ,  $c_2 = \frac{-9}{48}$ .

$$y(x) = \frac{5}{48} e^{7x} - \frac{9}{48} e^{3x} - \frac{1}{4} x e^{3x} + \frac{1}{12} e^x$$

- (c) (5 points) Find the general solution to the ODE

$$y^{(5)} + 8y^{(3)} + 16y' = 1 + (1 + e^x) \cos(2x).$$

**(NOTE:** For part (c), you do not need to evaluate the undetermined coefficients  $A, B, \dots$ ).

**Solution:** Here,  $P(r) = r^5 + 8r^3 + 16r = r(r^2 + 4)^2 = r(r + 2i)^2(r - 2i)^2$ .  
Therefore we have

$$y_c(x) = c_1 + c_2 \cos(2x) + c_3 \sin(2x) + c_4 x \cos(2x) + c_5 x \sin(2x),$$

$$y_p(x) = Ax + x^2(B \cos(2x) + C \sin(2x)) + De^x \cos(2x) + Fe^x \sin 2x,$$

and the general solution is of the form  $y = y_c + y_p$ .

6. Consider the following second order ODE for the function  $y(t)$ .

$$t^2 y'' + t y' + y = 0 \quad (t > 0).$$

- (a) (3 points) Verify that  $y_1(t) = \cos(\ln t)$  and  $y_2(t) = \sin(\ln t)$  are two linearly independent solutions to this ODE.

**Solution:**

$$t^2 y_1'' + t y_1' + y_1 = t^2(t^{-2} \sin(\ln t) + -t^{-2} \cos \ln t) + t(-t^{-1} \sin(\ln t)) + \cos(\ln t) = 0.$$

$$t^2 y_2'' + t y_2' + y_2 = t^2(-t^{-2} \cos(\ln t) + -t^{-2} \sin \ln t) + t(t^{-1} \cos(\ln t)) + \sin(\ln t) = 0.$$

These functions are obviously linearly independent.

- (b) (3 points) Using the substitutions  $x_1(t) = y(t)$  and  $x_2(t) = y'(t)$ , rewrite this ODE as an equivalent two-dimensional first order system of the form

$$\mathbf{x}' = P(t)\mathbf{x} \quad \text{where} \quad \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \& \quad P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

**Solution:**

$$x_1' = y' = x_2 \quad \& \quad x_2' = y'' = \frac{-y - t y'}{t^2} = -t^{-2} x_1 - t^{-1} x_2.$$

$$\implies \mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -t^{-2} & -t^{-1} \end{bmatrix} \mathbf{x}(t)$$

- (c) (3 points) Write down a fundamental matrix  $\Phi(t)$  for the system in part (b).  
(**Hint:** Use part (a).)

**Solution:**

$$\Phi(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1} \sin(\ln t) & t^{-1} \cos(\ln t) \end{bmatrix}.$$

(d) (6 points) Solve the *non-homogeneous* the initial value problem

$$\mathbf{x}' = P(t)\mathbf{x} + \begin{bmatrix} t^{-1} \\ 0 \end{bmatrix}; \quad \mathbf{x}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Solution:** We use the variation of parameters formula, noting that

$$\Phi(s)^{-1} = \begin{bmatrix} \cos(\ln s) & -s \sin(\ln s) \\ \sin(\ln s) & s \cos(\ln s) \end{bmatrix} \quad \& \quad \Phi(1)^{-1} = I.$$

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\Phi(1)^{-1}\mathbf{x}(1) + \Phi(t) \int_1^t \Phi(s)^{-1}\mathbf{f}(s)ds \\ &= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1} \sin(\ln t) & t^{-1} \cos(\ln t) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1} \sin(\ln t) & t^{-1} \cos(\ln t) \end{bmatrix} \int_1^t \begin{bmatrix} \cos(\ln s) & -s \sin(\ln s) \\ \sin(\ln s) & s \cos(\ln s) \end{bmatrix} \begin{bmatrix} s^{-1} \\ 0 \end{bmatrix} ds \\ &= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1} \sin(\ln t) & t^{-1} \cos(\ln t) \end{bmatrix} \int_1^t \begin{bmatrix} s^{-1} \cos(\ln s) \\ s^{-1} \sin(\ln s) \end{bmatrix} ds \\ &= \begin{bmatrix} \cos(\ln t) & \sin(\ln t) \\ -t^{-1} \sin(\ln t) & t^{-1} \cos(\ln t) \end{bmatrix} \begin{bmatrix} \sin(\ln t) \\ 1 - \cos(\ln t) \end{bmatrix} \\ &= \begin{bmatrix} \sin(\ln t) \\ t^{-1}(\cos(\ln t) - 1) \end{bmatrix} \end{aligned}$$

7. The matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}$$

has one eigenvalue  $\lambda$  with multiplicity 3.

(a) (6 points) Find the eigenvalue  $\lambda$  and all eigenvectors associated to  $\lambda$ .

**Solution:** The characteristic equation is:

$$\begin{aligned} 0 = P(\lambda) &= \det(A - \lambda I) = \det \begin{bmatrix} -1 - \lambda & 0 & 1 \\ 0 & 1 - \lambda & -4 \\ 0 & 1 & -3 - \lambda \end{bmatrix} \\ &= -(1 + \lambda)((1 - \lambda)(-3 - \lambda) + 4) - (1 + \lambda)(1 + 2\lambda + \lambda^2) \\ &= -(\lambda + 1)^3. \end{aligned}$$

Therefore  $\lambda = -1$  and if  $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$  is an eigenvector, then

$$(A - I)v = 0 \iff A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}.$$

(b) (2 points) What is the defect of this eigenvalue?

**Solution:** Since (up to scaling) there is only one eigenvector, the defect is  $3 - 1 = 2$ .

(c) (6 points) Find the general solution to the 3 dimensional linear system

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix} \mathbf{x}(t).$$

**Solution:** We must build a length three chain  $\{v_1, v_2, v_3\}$  of generalized eigenvectors based at an eigenvector  $v_1 = [1, 0, 0]^T$  (taking this as our choice for  $v_1$ ). Then we have for  $v_2 = [a, b, c]^T$ ,

$$v_1 = (A + I)v_2 \iff \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \implies c = 1 \text{ \& } b = 2.$$

Thus we may take  $v_2 = [0, 2, 1]^T$ . Next, if  $v_3 = [a, b, c]^T$ , then

$$v_2 = (A + I)v_3 \iff \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \implies c = 1 \text{ \& } b = 3.$$

Thus, we can take  $v_3 = [0, 3, 1]^T$ . The general solution is then

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t),$$

where

$$\mathbf{x}_1(t) = e^{-t}v_1, \quad \mathbf{x}_2(t) = e^{-t}(tv_1 + v_2), \quad \mathbf{x}_3(t) = e^{-t}\left(\frac{t^2}{2}v_1 + tv_2 + v_3\right).$$



*(Extra work space.)*